Lecture 10: Separable maps and LOCC

$\mathrm{Sep}(A:B)$

$$\rho_{AB} = \sum_i p_i \, \rho_{A,i} \otimes \rho_{B,i} \leftarrow \text{separable}$$

Choi - Jamiołkowski isomorphism:

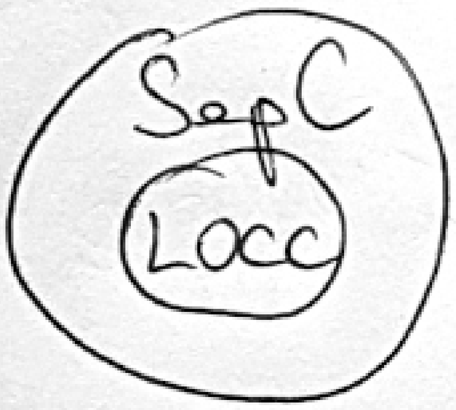
quantum channels $\cong$ quantum states

Entanglement cannot be increased by:

- local operations (unitary, isometry, local channel)
- classical communication ($A \longleftrightarrow B$ classical messages)

LOCC = local operations and classical communication

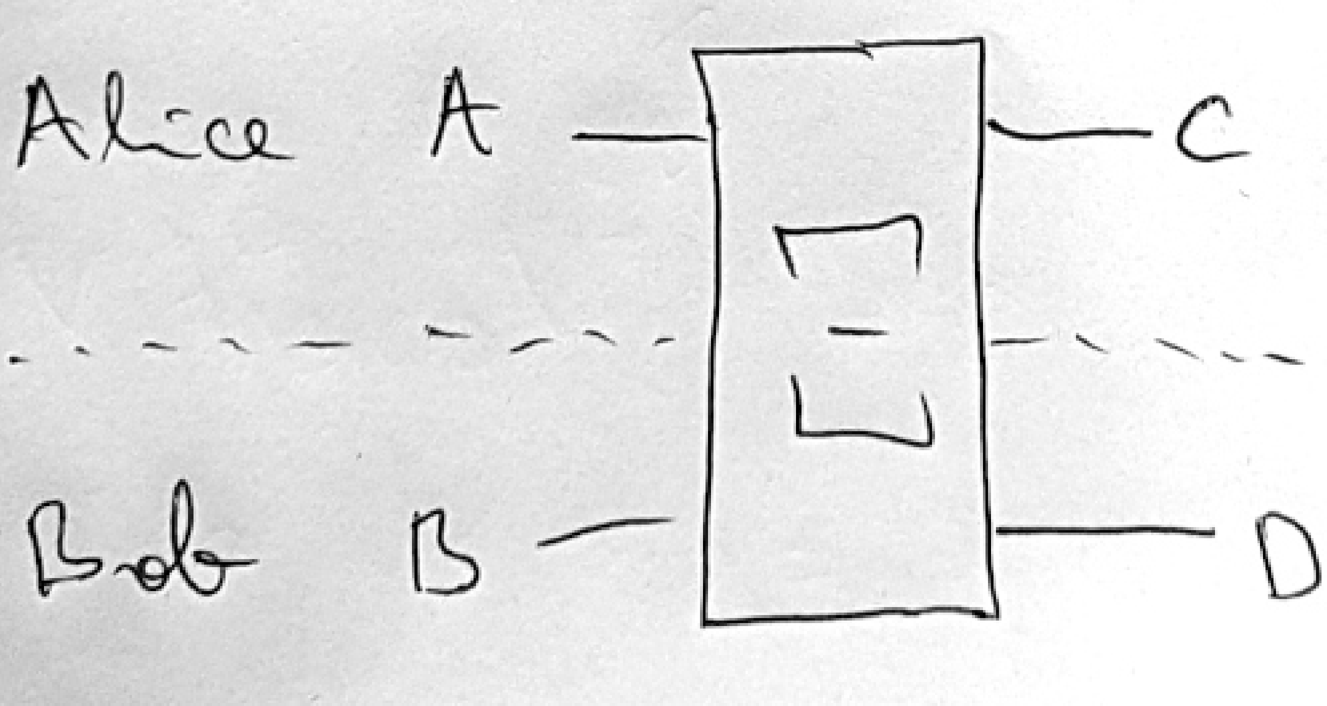
<u>Separable operations</u> - a relaxation of LOCC



$C(\mathcal{H}_A, \mathcal{H}_B)$ - quantum channels from $A \to B$

$CP(\mathcal{H}_A, \mathcal{H}_B)$ - completely positive maps from $A \to B$

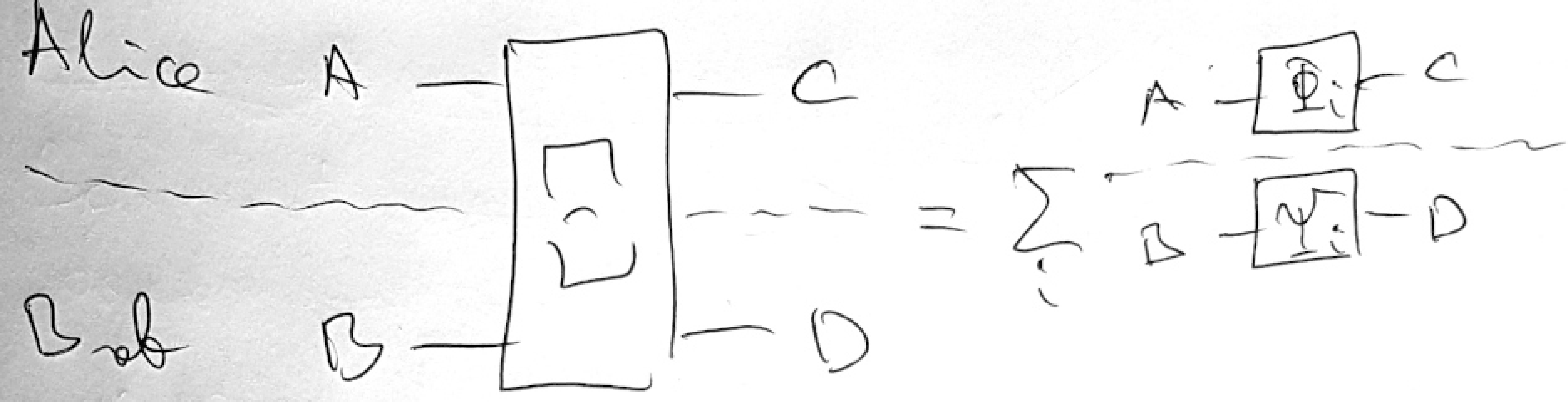
<u>Def</u> $\Xi \in CP(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{H}_C \otimes \mathcal{H}_D)$ is <u>separable</u> if



$$\Xi = \sum_i \Phi_i \otimes \Psi_i$$

$$\Phi_i \in CP(\mathcal{H}_A, \mathcal{H}_C)$$

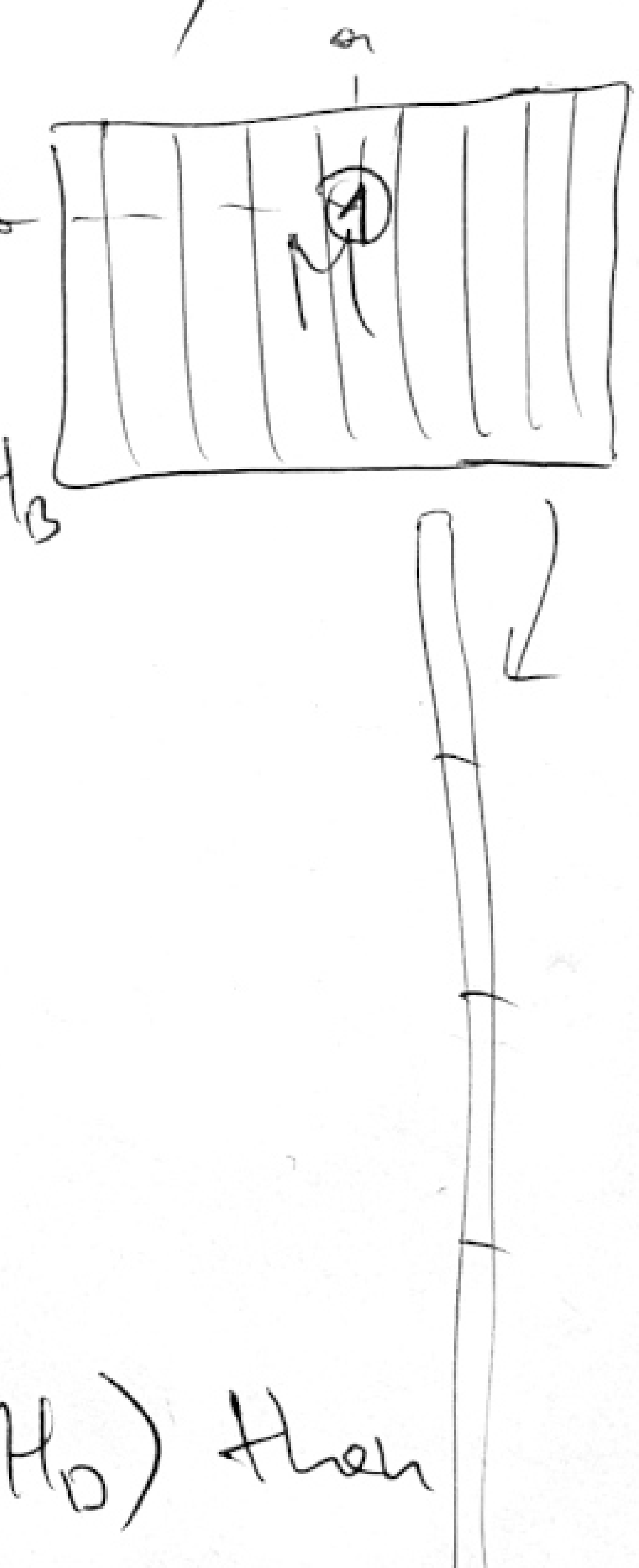
$$\Psi_i \in CP(\mathcal{H}_B, \mathcal{H}_D)$$



Notation: $\mathrm{SepCP}(\mathcal{H}_A, \mathcal{H}_C : \mathcal{H}_B, \mathcal{H}_D)$

sep. channels $\to \mathrm{SepC}(\text{——} , \text{——})$

Vectorization: $M \in L(\mathcal{H}_A, \mathcal{H}_B)$, $\mathcal{H}_A = \mathbb{C}^\Sigma$, $\mathcal{H}_B = \mathbb{C}^\Gamma$



$$|M_{AB}\rangle = \sum_{\substack{a\in\Sigma \\ b \in \Gamma}} \langle b|M|a\rangle\, |a,b\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$$

E.g., $M = |b\rangle\langle a| \to |M_{AB}\rangle = |a,b\rangle$

Vectorization identity:

$$(A \otimes B)|M\rangle = |BMA^T\rangle$$

Lemma $\Xi \in \mathrm{CP}(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{H}_C \otimes \mathcal{H}_D)$ then

$\Xi \in \mathrm{SepCP}(\mathcal{H}_A, \mathcal{H}_C : \mathcal{H}_B, \mathcal{H}_D) \leftarrow$ sep. superoperators

$$\Updownarrow$$

$V J^{\Xi}_{AB,CD} V^\dagger \in \mathrm{Sep}(\mathcal{H}_A \otimes \mathcal{H}_C : \mathcal{H}_B \otimes \mathcal{H}_D) \leftarrow$ sep. states

$V|a,b,c,d\rangle = |a,c,b,d\rangle$, $J^{\Xi} =$

Entanglement rank

Schmidt ~~rank~~ decomposition: $|\Psi_{AB}\rangle = \sum_{i=1}^{r} c_i |\alpha_{A,i}\rangle \otimes |\beta_{B,i}\rangle$ ← Schmidt rank

- product state: $r=1$
- max. ent. state in d dims: $r=d$

Can we extend this to mixed states?

Def $\rho_{AB} \in \mathrm{Ent}_r(\mathcal{H}_A : \mathcal{H}_B) \subseteq \mathrm{PSD}(\mathcal{H}_A \otimes \mathcal{H}_B)$ if

(has ent. rank $\leq r$)

$$\rho_{AB} = \sum_x |\Psi_{AB,x}\rangle\langle\Psi_{AB,x}|$$

where each $|\Psi_{AB,x}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ has Schmidt rank $\leq r$. The ent. rank of ρ_{AB} is the smallest r s.t. $\rho_{AB} \in \mathrm{Ent}_r(\mathcal{H}_A : \mathcal{H}_B)$.

Ex. $\mathrm{Sep} = \mathrm{Ent}_1 \subset \cdots \subset \mathrm{Ent}_r \subset \cdots \subset \mathrm{Ent}_n = \mathrm{PSD}$

$|\Psi_{AB}\rangle = |\alpha_A\rangle \otimes |\beta_B\rangle$ $\qquad n = \min\{\dim \mathcal{H}_A, \dim \mathcal{H}_B\}$

Theorem Separable maps cannot increase entanglement rank.

Proof $\rho_{AB} = \sum_y |M_y\rangle\langle M_y|$, $\mathrm{rank}(M_y) \leq r$

$$\Xi(\rho) = \sum_x \sum_y (A_x \otimes B_x)|M_y\rangle\langle M_y|(A_x \otimes B_x)^\dagger$$

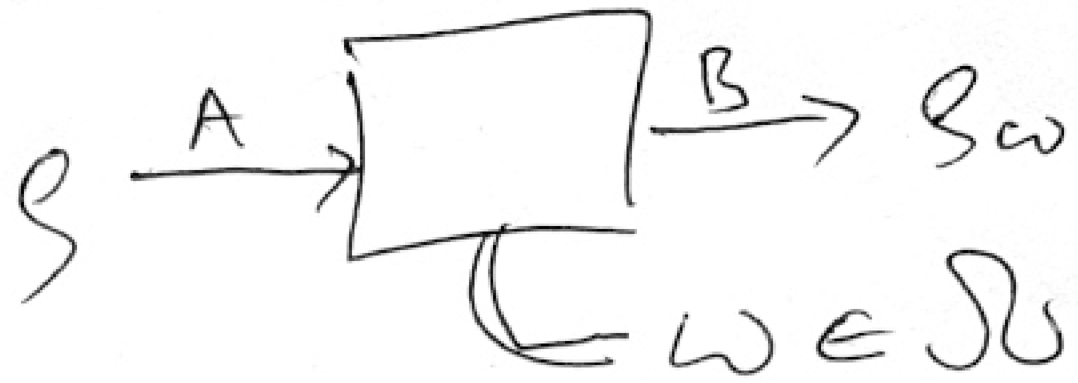
$$= \sum_x \sum_y |B_x M_y A_x^T\rangle\langle B_x M_y A_x^T|$$

$$\mathrm{rank}(B_x M_y A_x^T) \leq \mathrm{rank}(M_y)$$

Cor. Separable maps cannot map a sep. state to an entangled state.

Same true for LOCC $\subseteq$ SepC.

Instrument



$\{\Phi_\omega : \omega\in\Omega\} \subset CP(\mathcal{H}_A,\mathcal{H}_B)$ s.t. $\sum_{\omega\in\Omega}\Phi_\omega \in C(\mathcal{H}_A,\mathcal{H}_B)$

(Contrast with measurements: $\mu : \Omega \to PSD(\mathcal{H})$, can think as $\{\mu(\omega) : \omega\in\Omega\}$.)

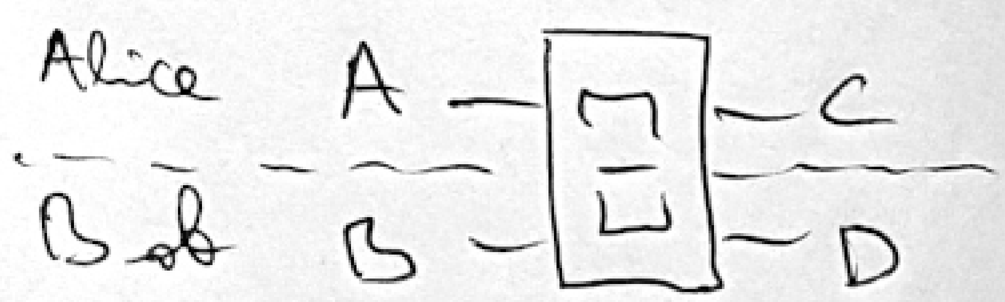
If we apply this instrument to $\rho\in D(\mathcal{H}_A)$ we get outcome $\omega\in\Omega$ w.p. $Tr[\Phi_\omega(\rho)]$ and state becomes

$$\rho_\omega = \frac{\Phi_\omega[\rho]}{Tr[\Phi_\omega[\rho]]} \in D(\mathcal{H}_B)$$

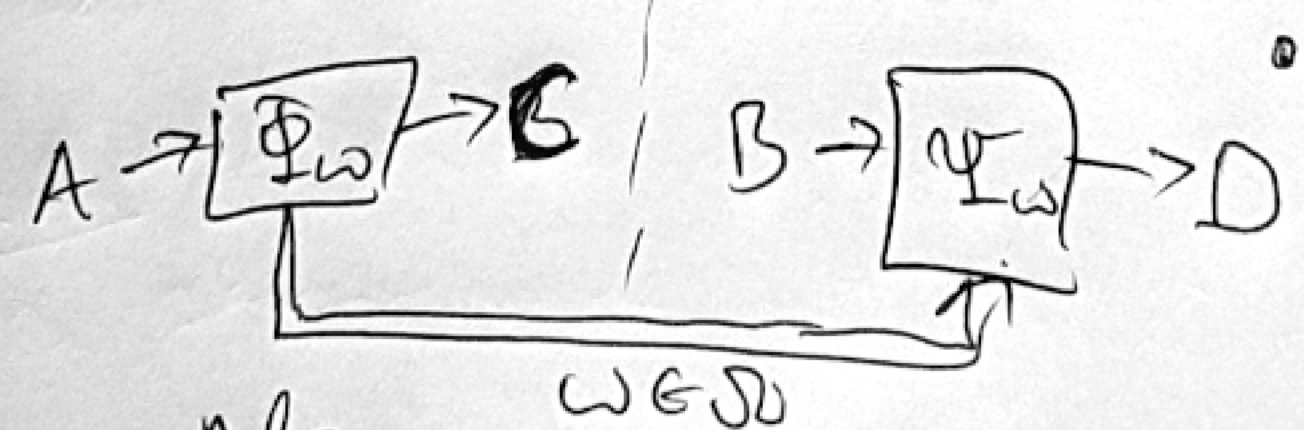
Def Alice $\rightleftarrows$ Bob

• one-way right LOCC

$$\Xi = \sum_{\omega\in\Omega} \Phi_\omega \otimes \Psi_\omega$$



$\Xi \in C(\mathcal{H}_A\otimes\mathcal{H}_B, \mathcal{H}_C\otimes\mathcal{H}_D)$



• where $\{\Phi_\omega : \omega\in\Omega\}$ is an instrument on Alice's side,

• $\Psi_\omega \in C(\mathcal{H}_B,\mathcal{H}_D)$ on Bob's side

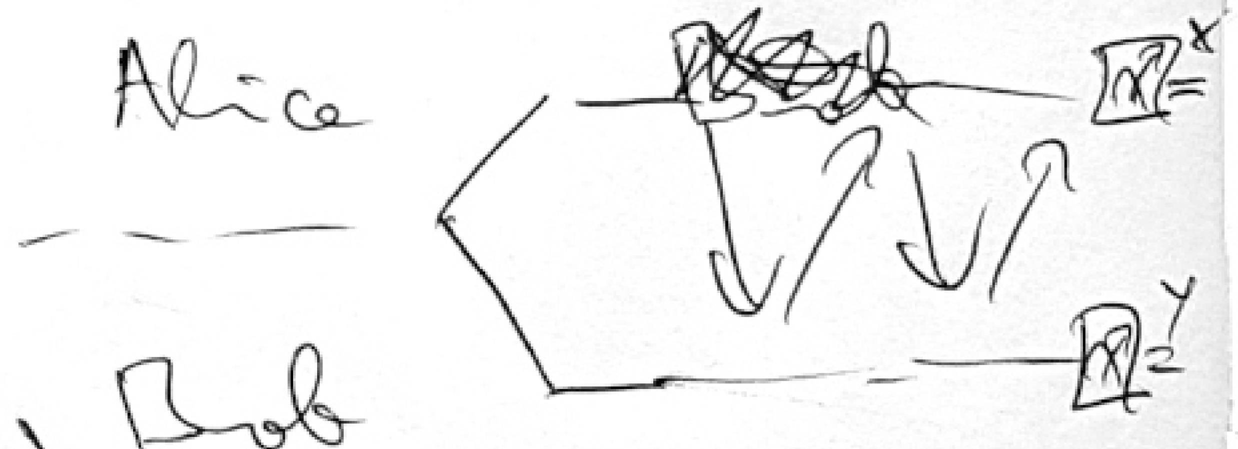
- <u>one-way left LOCC</u> is similar, with Alice and Bob exchanged.
- Ξ is an LOCC channel if it is a finite composition of the above.

Notation: $\mathrm{LOCC}(\underbrace{\mathcal{H}_A \overset{\text{time}}{\curvearrowright} \mathcal{H}_C}_{\text{Alice}} : \underbrace{\mathcal{H}_B \overset{\text{time}}{\curvearrowright} \mathcal{H}_D}_{\text{Bob}})$

Exercise:

$$\mathrm{LOCC}(\mathcal{H}_A, \mathcal{H}_C : \mathcal{H}_B, \mathcal{H}_D)$$
$$\subseteq \mathrm{SepC}(\text{——— " ———})$$

<u>Sep. and LOCC measurements</u>



<u>Def</u> $\mu : \Omega \to \mathrm{PSD}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\mathcal{H}_X = \mathcal{H}_Y = \mathbb{C}^{\Omega}$, then let $\Phi_\mu \in C(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{H}_X \otimes \mathcal{H}_Y)$ be the quantum-to-classical channel

$$\Phi_\mu(X_{AB}) = \sum_{\omega \in \Omega} \mathrm{Tr}\left[\mu(\omega) X_{AB}\right] \cdot \overset{\text{Alice}}{|\omega\rangle\langle\omega|_X} \otimes \overset{\text{Bob}}{|\omega\rangle\langle\omega|}$$

Then μ is separable/LOCC if Φ_μ is sep./LOCC.

<u>Lem</u> A measurement $\mu : \Omega \to \mathrm{PSD}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable iff $\mu(\omega) \in \mathrm{Sep}(\mathcal{H}_A : \mathcal{H}_B)$

Def (one-way right (left LOCC measurement)

$\mu : \Omega \to PSD(\mathcal{H}_A \otimes \mathcal{H}_B)$ is one-way right LOCC measurement if

$$\text{Alice} \xrightarrow[\gamma \in \Gamma]{\text{message}} \text{Bob}$$

$$\mu(\omega)_{AB} = \sum_{\gamma \in \Gamma} \nu(\gamma)_A \otimes \pi_\gamma(\omega)_B$$

where $\pi_\gamma : \Omega \to PSD(\mathcal{H}_B)$ is a measurement on Bob's side for every $\gamma \in \Gamma$.

and $\nu : \Gamma \to PSD(\mathcal{H}_A)$ is a measurement on Alice's side.

- Similarly for one-way left LOCC.

Theorem Let $|\Psi_0\rangle_{AB}, |\Psi_1\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, assume orthogonal (e.g., two Bell states). Then there exists a perfect one-way right measurement $\mu : \{0,1\} \to PSD(\mathcal{H}_A \otimes \mathcal{H}_B)$ s.t.

$$1 = \langle \Psi_0 | \mu(0) | \Psi_0 \rangle = \langle \Psi_1 | \mu(1) | \Psi_1 \rangle .$$