Quantum Information Theory, Spring 2020

Practice problem set #9

You do **not** have to hand in these exercises, they are for your practice only.

1. **Warmup:**

   (a) Show that every classical state ρ_{XY} is separable.

   (b) Let $|\psi_{AB}\rangle$ be a pure state. Show that the following are equivalent: (i) $|\psi_{AB}\rangle$ is separable, (ii) its Schmidt rank is one, (iii) its entanglement entropy is zero. Recall from Lemma 2.12 that the Schmidt rank is the number of non-zero coefficients in the Schmidt decomposition.

   (c) In the lecture we defined *separable operators* to be those that can be written as

   $$\sum_i P_{A,i} \otimes Q_{B,i}$$

   where $P_{A,i}$ and $Q_{B,i}$ are positive semidefinite. Show that the restriction of this definition to density matrices coincides with the definition of separable states from the lecture. Show that restricting this further to pure states also coincides with the definition from the lecture.

   (d) Recall that the four *Bell states* are defined by

   $$|\Phi^{zx}\rangle = (Z^z X^x \otimes I)|\Phi^+\rangle$$

   where $z, x \in \{0, 1\}$ and $|\Phi^+\rangle$ is the canonical two-qubit maximally entangled state. Verify that

   $$|\Phi^{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\Phi^{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
   $$|\Phi^{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \qquad |\Phi^{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

   Verify also that

   $$|\Phi^{zx}\rangle = (I \otimes X^x Z^z)|\Phi^+\rangle.$$

2. **Maximally entangled state tricks:** Let $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^\Sigma$ and

   $$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{|\Sigma|}} \sum_{x \in \Sigma} |x\rangle \otimes |x\rangle.$$

   a maximally entangled state.

   (a) Show that, for any $M \in L(\mathcal{H}_A)$,

   $$(M \otimes I)|\Phi^+\rangle = (I \otimes M^T)|\Phi^+\rangle.$$

   (b) Show that for $M, N \in L(\mathcal{H}_B)$

   $$\mathrm{Tr}(M^\dagger N) = |\Sigma| \langle \Phi^+ | \overline{M} \otimes N | \Phi^+ \rangle.$$

3. **Two-qubit pure states (product vs entangled):** Let

$$|\psi\rangle = \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix} \in \mathbb{C}^4.$$

be an arbitrary pure state on two qubits. Define

$$\Delta(|\psi\rangle) = \psi_{00}\psi_{11} - \psi_{01}\psi_{10}.$$

The goal of this exercise is to show that $\Delta(|\psi\rangle) = 0$ if and only if $|\psi\rangle$ is a product state.

(a) Assume that $|\psi\rangle$ is a product state, i.e., $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$, for some single-qubit states

$$|\alpha\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}, \qquad\qquad |\beta\rangle = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}.$$

Show that in such case $\Delta(|\psi\rangle) = 0$.

(b) Conversely, let $|\psi\rangle$ be an arbitrary two-qubit state and assume that $\Delta(|\psi\rangle) = 0$. Find two single-qubit states $|\alpha\rangle$ and $|\beta\rangle$ such that $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$.

The quantity $\Delta(|\psi\rangle)$ can not only be used to determine if a pure two-qubit state is entangled or not but is also a meaningful measure of the amount of entanglement.

4. **Pauli matrices and the swap:** Let $\Sigma = \{0, 1\}$ and $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^\Sigma$. The two-qubit *swap operation* $W \in \mathrm{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is defined on the computational basis states as follows:

$$W|a, b\rangle = |b, a\rangle,$$

for all $a, b \in \{0, 1\}$. Recall that the four *Pauli matrices* are

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Verify that

$$W = \frac{1}{2}(I \otimes I + X \otimes X + Y \otimes Y + Z \otimes Z).$$

(b) Verify that

$$W = \frac{1}{2} \sum_{z,x \in \{0,1\}} Z^z X^x \otimes X^x Z^z.$$