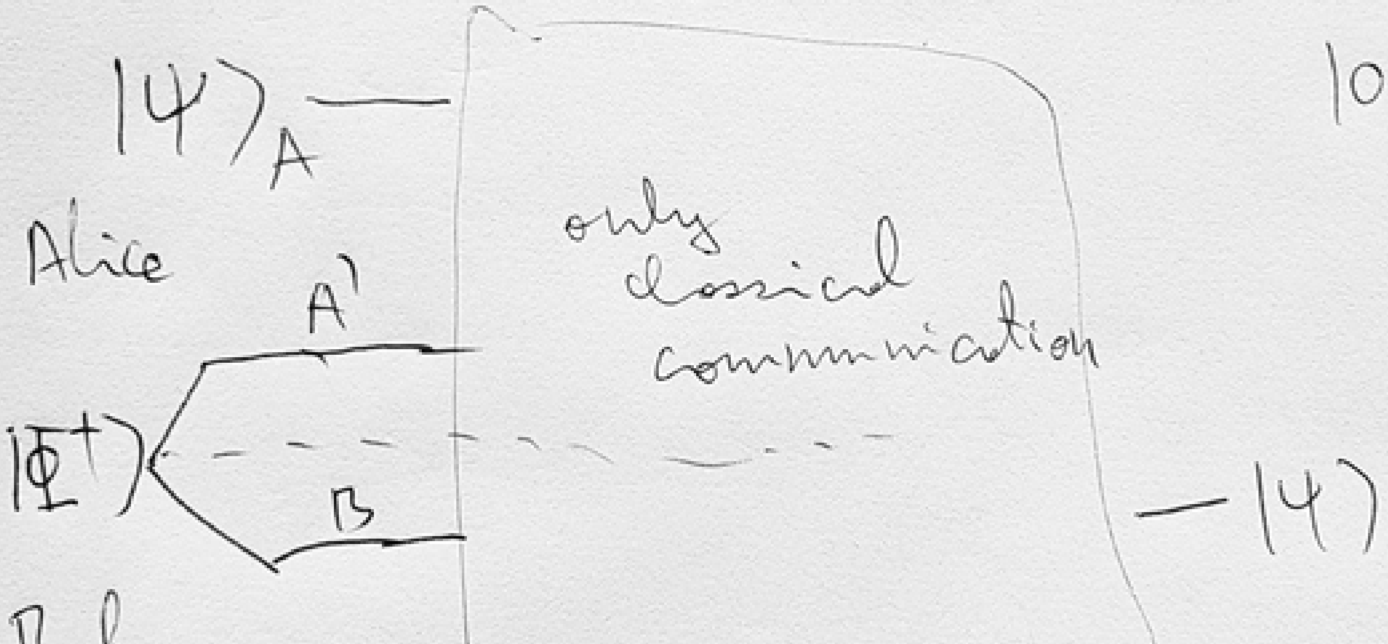
# Lecture 9: Entanglement

Alice $\qquad\qquad$ Bob

$$\rho_{AZ} = 2^n \times 2^n \xrightarrow[\text{message}]{\text{classical}} \rho_{ij} \ \forall i,j$$

Zoe


We let Alice and Bob to meet beforehand and share $|\Phi^+_{A'B}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.



$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a, b, c, d \in \mathbb{C}$

Pauli matrices:

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\langle \sigma_i, \sigma_j \rangle = \mathrm{Tr}[\sigma_i^\dagger \sigma_j] = 2\delta_{ij}$

$ZX = -XZ = iY$

$Z^z X^x, \quad z, x \in \{0,1\}$ $I^2 = X^2 = Y^2 = Z^2 = I$

$Z^0 X^0 = I \quad Z^0 X^1 = X \quad Z^1 X^0 = Z \quad Z^1 X^1 = iY$

Swap: $W|a, b\rangle = |b, a\rangle \quad \forall a, b \in \{0,1\}$

$\Leftrightarrow W|\psi\rangle \otimes |\varphi\rangle = |\varphi\rangle \otimes |\psi\rangle \quad \forall |\psi\rangle, |\varphi\rangle \in \mathbb{C}^2$

$$W = \frac{1}{2}(I \otimes I + X \otimes X + Y \otimes Y + Z \otimes Z)$$

$$= \frac{1}{2} \sum_{z, x \in \{0,1\}} (Z^z X^x \otimes X^x Z^z)$$

Bell states: $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\mathrm{span}\{|\Phi^{zx}\rangle : x, z \in \{0,1\}\} = \mathbb{C}^4$

$|\Phi^{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad |\Phi^{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$|\Phi^{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \quad |\Phi^{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

$$|\Phi^{zx}\rangle = (Z^z X^x \otimes I)|\Phi^{00}\rangle \quad z, x \in \{0,1\}$$
$$= (I \otimes X^x Z^z)|\Phi^{00}\rangle$$

Preparation/unpreparation:

$$|z\rangle \otimes |x\rangle$$

$$|\Phi^{zx}\rangle = \mathrm{CNOT} \cdot (H \otimes I) \cdot |z, x\rangle$$

$$|z, x\rangle = (H \otimes I) \cdot \mathrm{CNOT} \cdot |\Phi^{zx}\rangle$$

$\xleftarrow{\text{time}}$

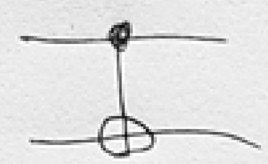
$$\mathrm{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

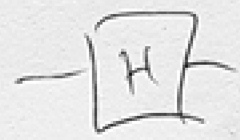
(controlled-NOT)

(Hadamard)

$$= \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathrm{CNOT}^2 = I \qquad H^2 = I$$

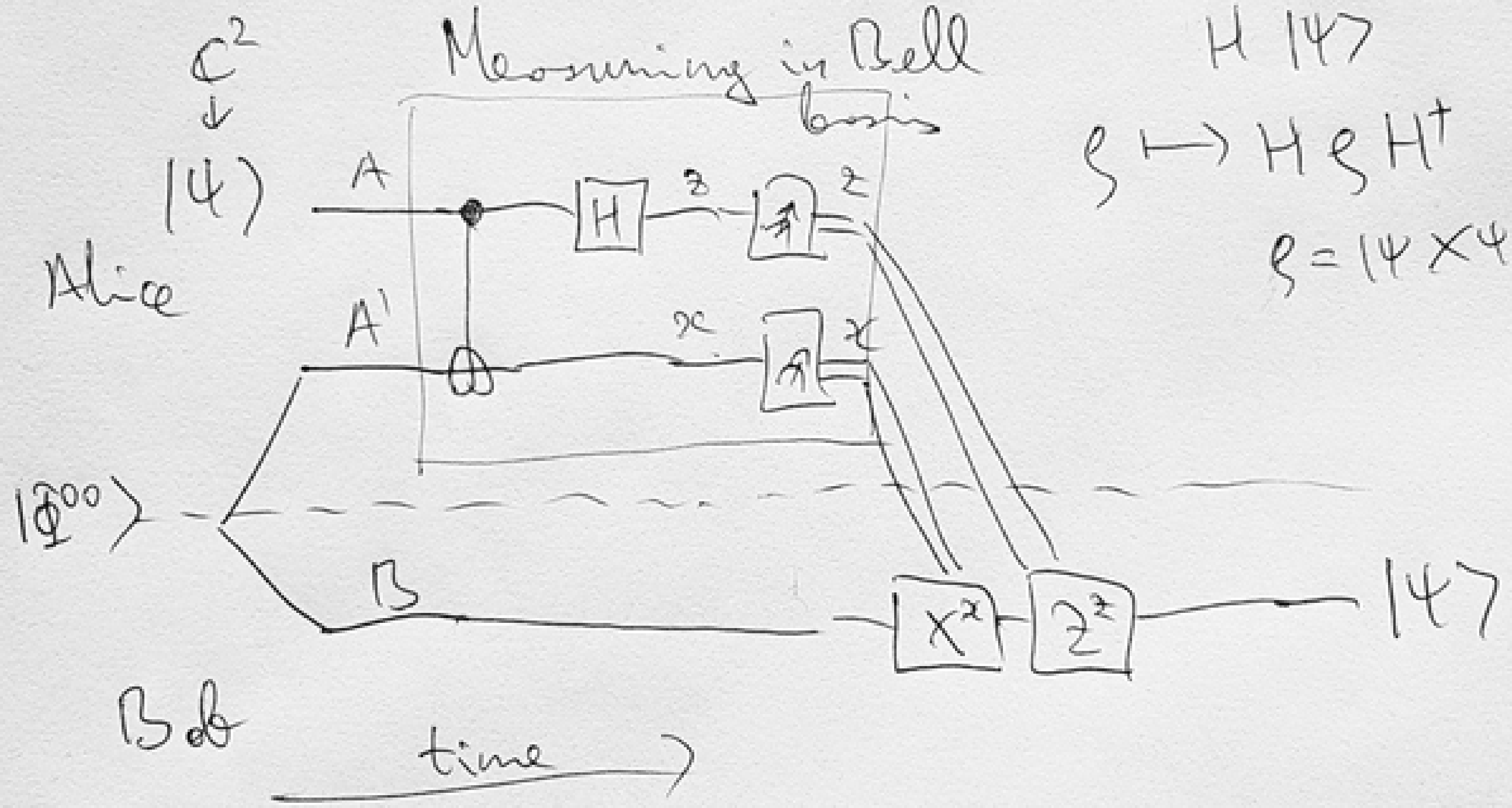




$H|\psi\rangle$

$\rho \longmapsto H \rho H^\dagger$

$\rho = |\psi\rangle\langle\psi|$




$$\forall |\psi\rangle \in \mathbb{C}^2 : \quad |\psi\rangle_A \otimes |\Phi^{00}\rangle_{A'B} = \frac{1}{2} \sum_{z,x \in \{0,1\}} |\Phi^{zx}\rangle_{AA'} \otimes X^x Z^z |\psi\rangle_B$$

$\rho_{AB}$ measure A with

$\mu_A : \Omega \to PSD(\mathcal{H}_A)$

<u>Axiom</u> Probability: $\omega \in \Omega$

$$p_\omega = \mathrm{Tr}\left[\rho_{AB} \cdot (\mu_A(\omega) \otimes I_B)\right]$$

$$\rho_{B,\omega} = \frac{1}{p_\omega} \mathrm{Tr}_A\left[\rho_{AB} \cdot (\mu_A(\omega) \otimes I_B)\right]$$

$$\underset{0/1}{O} \qquad \underset{0/1}{O}$$

$$AB: \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \overset{AB}{\begin{pmatrix} p_{00} \\ p_{01} \\ p_{10} \\ p_{11} \end{pmatrix}} \leftarrow \text{classical distribution}$$

$$\overset{?}{=} p' \otimes p''$$

$$\overset{?}{\neq} p' \otimes p''$$

Prob (0 from Alice) $= p_{00} + p_{01} = p_0$

Bob: $\left.\begin{array}{lll} 0 & \text{w.p.} & \frac{p_{00}}{p_0} \\ 1 & \text{w.p.} & \frac{p_{01}}{p_0} \end{array}\right\} = \frac{1}{p_0}\begin{pmatrix} p_{00} \\ p_{01} \end{pmatrix}$

"reduced"
state if Alice has 0

"Dual" of teleportation is called
superdense coding.
(2 bits into 1 qubit)
Catch: share $|\Phi^{00}\rangle_{AB}$

Resource inequalities:

~~ebit~~ $|\Phi^+\rangle$
1 bit of classical com.

teleportation: $\text{ebit} + 2[c \to c] \geq [q \to q]$

1 bit of quantum comm.

Superdense coding:

$$\underline{\text{ebit}} + [q \to q] \geq 2[c \to c]$$

Entanglement

Define by not being separable.

~~Def~~ $\mathcal{H}_A, \mathcal{H}_B$

$M_{AB} \in PSD(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable if

$$\Updownarrow \quad M_{AB} = \sum_i P_{A_i}^{(i)} \otimes Q_B^{(i)}, \quad P_A^{(i)} \in PSD(\mathcal{H}_A)$$
$$Q_B^{(i)} \in PSD(\mathcal{H}_B)$$

$\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable if

$$\Updownarrow \quad \rho_{AB} = \sum_i p_i \, \rho_A^{(i)} \otimes \rho_B^{(i)} \quad \rho_A \in D(\mathcal{H}_A)$$
$$\rho_B \in D(\mathcal{H}_B)$$

↑ prob. dist.

$\Downarrow$

$|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is separable if

or product

$$|\psi\rangle_{AB} = |\alpha_A\rangle \otimes |\beta_B\rangle \qquad |\alpha_A\rangle \in \mathcal{H}_A \quad |\beta_B\rangle \in \mathcal{H}_B$$

(You don't get $|\psi\rangle_{AB} = \sum_i c_i |\alpha^i_A\rangle \otimes |\beta^i_B\rangle$
because any state can be written like this)
Schmidt decomp.

Ex

$$\frac{1}{2}\overbrace{|00\rangle\langle 00|}^{|0\rangle\otimes|0\rangle} + \frac{1}{2}|11\rangle\langle 11| = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$$ is classical!

$$= \frac{1}{2}|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + \frac{1}{2}|1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B$$

$|0\rangle\langle 0|, |1\rangle\langle 1| \in D(\mathbb{C}^2)$

$\Rightarrow$ is separable

Ex Any classical state is separable.

If $|\psi\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B$ then $(U_A \otimes V_B)|\psi\rangle_{AB}$

$$= (U|\alpha\rangle)_A \otimes (V|\beta\rangle)_B$$

Local operations do not increase (create) entanglement).

$$(U\otimes V)\,\rho_{AB}\,(U\otimes V)^\dagger = \sum_i p_i \left(U\rho^{(i)}_A U^\dagger\right) \otimes \left(V\rho^{(i)}_B V^\dagger\right)$$

# Entanglement entropy

$|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \qquad H(A)_\psi = H(B)_\psi$

entropy of the reduced state

$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|_{AB}) \qquad H(A) := H(\rho_A)$

Exercise: $|\psi_{AB}\rangle$ ~~entangled~~ separable (product)

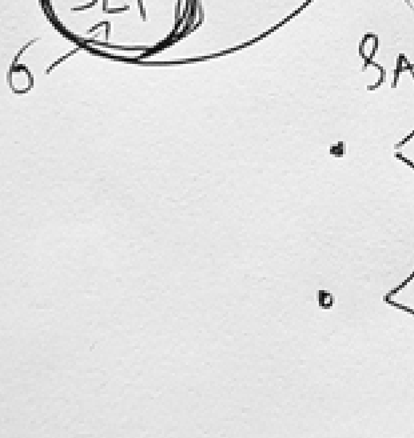
$\Leftrightarrow$ — " — has Schmidt rank 1

$\Leftrightarrow H(A) = H(B) = 0$

$\Leftrightarrow |\psi_{AB}\rangle = |\alpha_A\rangle \otimes |\beta_B\rangle$

# Criterion for entanglement

Set of all sep. states is convex & compact
(mixed) (closed and bounded)




entangled

hyperplane — entanglement witness
$H^\dagger = H$

$\rho_{AB}$ is entangled

- $\langle \rho, H\rangle < 0$
- $\langle \sigma, H\rangle > 0$ for all $\sigma \in \mathrm{SEP}$

$\langle A, B\rangle := \mathrm{Tr}(A^\dagger B)$

Thm (Horodecki)

$\rho_{AB}$ is separable

$\iff (\Psi_A \otimes I_B)(\rho_{AB}) \geq 0$ PSD

$\forall \Psi_A$ that are unital and positive

$\Psi(I) = I$

$P \geq 0 \Rightarrow \Psi(P) \geq 0$

Typical use case:

$\Psi_A(M) := M^T$ ← the transpose

Ex

$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is it entangled?

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\Psi \otimes I} \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \not\geq 0$$

$\Rightarrow |\Phi^+\rangle$ is entangled $\qquad \pm 1$ eigenvalues.