Quantum Information Theory, Spring 2020

Practice problem set #8

You do **not** have to hand in these exercises, they are for your practice only.

1. **Warmup:**
   1. Show that, if ρ and σ are both pure states, $D(\rho\|\sigma) \in \{0, \infty\}$.
   2. Find a state ρ and a channel Φ such that $H(\Phi[\rho]) < H(\rho)$.
   3. Compute the relative entropy $D(\rho\|\sigma)$ for $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ and $\sigma = \frac{1}{4}|+\rangle\langle +| + \frac{3}{4}|-\rangle\langle -|$.

2. **Matrix logarithm:** Recall that the logarithm of a positive definite operator with eigendecomposition $Q = \sum_i \lambda_i |e_i\rangle\langle e_i|$ is defined as $\log(Q) = \sum_i \log(\lambda_i)|e_i\rangle\langle e_i|$ (as always, our logarithms are to base 2). Verify the following properties:
   1. $\log(cI) = \log(c)I$ for every $c \geqslant 0$.
   2. $\log(Q \otimes R) = \log(Q) \otimes I_B + I_A \otimes \log(R)$ for all positive definite $Q \in L(\mathcal{H}_A)$, $R \in L(\mathcal{H}_B)$.
   3. $\log(\sum_{x\in\Sigma} p_x |x\rangle\langle x| \otimes \rho_x) = \sum_{x\in\Sigma} \log(p_x)|x\rangle\langle x| \otimes I_B + \sum_{x\in\Sigma} |x\rangle\langle x| \otimes \log(\rho_x)$ for every ensemble $\{p_x, \rho_x\}_{x\in\Sigma}$ of positive definite operators $\rho_x \in D(\mathcal{H}_B)$.

   Warning: It is in general not true that $\log(QR) = \log(Q) + \log(R)$*!*

3. **From relative entropy to entropy and mutual information:** Use Problem 2 to verify the following claims from class:
   1. $D(\rho\|\frac{I}{d}) = \log d - H(\rho)$ for every $\rho \in D(\mathcal{H})$, where $d = \dim \mathcal{H}$.
   2. $D(\rho_{AB}\|\rho_A \otimes \rho_B) = I(A:B)_{\rho_{AB}}$ for every $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\rho_A = \mathrm{Tr}_B[\rho_{AB}]$ and $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$. You may assume that all three operators are positive definite.

4. **Entropy and ensembles:** In this problem, you will prove the upper bound on the Holevo information that we discussed in class: For every ensemble $\{p_x, \rho_x\}$,

   $$\chi(\{p_x, \rho_x\}) \leqslant H(p) \quad \text{or, equivalently,} \quad H(\sum_x p_x \rho_x) \leqslant H(p) + \sum_x p_x H(\rho_x).$$

   Moreover, show that equality holds if and only if the ρ_x with $p_x > 0$ have pairwise orthogonal image.

   In terms of the cq-state corresponding to the ensemble, the above inequality can also be written as $H(XB) \geqslant H(B)$. This confirms our claim in Lecture 7. In Homework Problem 6.4 you showed that $H(XB) \geqslant H(X)$, but since the situation is not symmetric (X is classical but B is not) we now need a different argument.

   1. First prove these claims assuming that each ρ_x is a pure state, i.e., $\rho_x = |\psi_x\rangle\langle\psi_x|$.

      Hint: Consider the pure state $|\Phi\rangle = \sum_x \sqrt{p_x}|x\rangle \otimes |\psi_x\rangle$ *and compare the entropy of the first system before and after measuring in the standard basis.*
   2. Now prove the claims for general ρ_x.

      Hint: Apply part (a) to a suitable ensemble obtained from the eigendecompositions of the ρ_x.