Quantum Information Theory, Spring 2020

Homework problem set #7 **due March 23, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(2 points)* **Quantum mutual information:** From class, we know that $I(A : B) \leqslant 2 \log d$ for every state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^d$. Show that $I(A : B) = 2 \log d$ if and only if ρ_{AB} is a pure state with $\rho_A = \rho_B = I/d$ (such states are called *maximally entangled*). Write down the Schmidt decomposition of a general state of this form.

   Hint: In the exercise class you gave simple proof of the above inequality.

2. *(2 points)* **Classical mutual information:** From class, we know that $I(X : Y) \leqslant \log d$ for every distribution $p_{XY} \in P(\Sigma_X \times \Sigma_Y)$ with $|\Sigma_X| = |\Sigma_Y| = d$. Show that $I(X : Y) = \log d$ if and only if $p_{XY}(x, y) = \frac{1}{d}\delta_{f(x),y}$ for a bijection $f \colon \Sigma_X \to \Sigma_Y$ (such p_{XY} are called *maximally correlated*).

   Hint: In the exercise class you characterized the probability distributions with $H(XY) = H(X)$.

3. *(8 points)* **Entropic uncertainty relation:** Here you can prove another uncertainty relation. Let $\rho \in D(\mathbb{C}^2)$ and denote by p_{Std} and p_{Had} the probability distributions of outcomes when measuring ρ in the standard basis and Hadamard basis, respectively. You will show:

$$H(p_{\text{Std}}) + H(p_{\text{Had}}) \geqslant H(\rho) + 1 \tag{1}$$

   (a) Why is it appropriate to call (1) an *uncertainty relation*?
   (b) Find a state ρ for which the uncertainty relation is saturated (i.e., an equality).

   To start, recall the Pauli matrices $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   (c) Verify that $\frac{1}{2}(\rho + Z\rho Z) = \begin{pmatrix} \langle 0|\rho|0\rangle & 0 \\ 0 & \langle 1|\rho|1\rangle \end{pmatrix}$ and deduce that $H(p_{\text{Std}}) = H(\frac{1}{2}(\rho + Z\rho Z))$.
   (d) Show that, similarly, $H(p_{\text{Had}}) = H(\frac{1}{2}(\rho + X\rho X))$. *Hint:* $|\pm\rangle$ *is the eigenbasis of* X.

   Now consider the following three-qubit state,

$$\omega_{ABC} = \frac{1}{4} \sum_{a=0}^{1} \sum_{b=0}^{1} |a\rangle\langle a| \otimes |b\rangle\langle b| \otimes X^a Z^b \rho Z^b X^a,$$

   where we denote $X^0 = I$, $X^1 = X$, $Z^0 = I$, $Z^1 = Z$. Note that subsystems A & B are classical.

   (e) Show that $H(ABC) = 2 + H(\rho)$. Use parts (c) and (d) to verify that $H(AC) = 1 + H(p_{\text{Std}})$, $H(BC) = 1 + H(p_{\text{Had}})$, and $H(C) = 1$ in state ω_{ABC}.

   Hint: Use the formula for the entropy of classical-quantum states that you proved last week.
   (f) Use part (e) and the strong subadditivity inequality to deduce (1).

4. *(2 bonus points)* **Practice:** In this problem, you can explore the properties of typical subspaces. Consider the qubit state $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

   (a) Compute the largest eigenvalue λ as well as the von Neumann entropy $H(\rho)$ of ρ.

   (b) Plot the following functions of $k \in \{0, 1, \ldots, n\}$ for $n = 100$ as well as for $n = 1000$:

   $$d(k) = \binom{n}{k}, \quad r(k) = \frac{1}{n} \log \binom{n}{k}, \quad q(k) = \binom{n}{k} \lambda^k (1-\lambda)^{n-k}$$

   (c) Plot the following functions of $n \in \{1, \ldots, 1000\}$ for $\varepsilon = 0.1$ as well as for $\varepsilon = 0.01$:

   $$r(n) = \frac{1}{n} \log \dim S_{n,\varepsilon}, \quad p(n) = \mathrm{Tr}[\Pi_{n,\varepsilon} \rho^{\otimes n}],$$

   where $\Pi_{n,\varepsilon}$ denotes the orthogonal projection onto the typical subspace $S_{n,\varepsilon}$ of ρ.