Recall:

# Proof of Schumacher's Theorem

**Def:** $(n, R, \delta)$ – quantum code for $\rho \in D(\mathcal{H}_A)$: quantum channels

$$\mathcal{E} \in C(\mathcal{H}_A^{\otimes n}, (\mathbb{C}^2)^{\otimes \lfloor Rn \rfloor}) \quad \text{and} \quad \mathcal{D} \in C((\mathbb{C}^2)^{\otimes \lfloor Rn \rfloor}, \mathcal{H}_A^{\otimes n})$$

s.th. $F(\mathcal{D} \circ \mathcal{E}, \rho^{\otimes n}) \geq 1 - \delta$

channel fidelity $F(T_A, \rho_A) = \inf \{ F((T_A \otimes I_B)[\sigma_{AB}]) : \mathcal{H}_B, \sigma_{AB} \text{ s.th. } \sigma_A = \rho_A \}$

$$= \sqrt{\sum_k \operatorname{tr}[Z_k \rho]^2} \quad \text{if } \{Z_k\} \text{ kraus ops for } T_A$$

**Schumacher's Theorem:** Let $0 < \delta < 1$:

① If $R > H(\rho)$: $\exists n_0 : \forall n \geq n_0 : \exists (n, R, \delta)$ – q code

② If $R < H(\rho)$: $\exists n_0 : \forall n \geq n_0 : \nexists (n, R, \delta)$ – q code

key tool:

**Typical projectors:** $\Pi_{n,\varepsilon}$ onto typical subspaces $S_{n,\varepsilon} = S_{n,\varepsilon}(\rho) \subseteq \mathcal{H}_A^{\otimes n}$

Properties: ⓪ Eigenvalues of $\Pi_{n,\varepsilon} \rho^{\otimes n} \Pi_{n,\varepsilon}$ are in $2^{-n(H(\rho) \pm \varepsilon)}$

① $\operatorname{rk} \Pi_{n,\varepsilon} = \dim S_{n,\varepsilon} \leq 2^{n(H(\rho)+\varepsilon)}$

② $\operatorname{tr}[\Pi_{n,\varepsilon} \rho^{\otimes n}] \longrightarrow 1$

Proof of Schumacher's Theorem, part ①: Choose $\varepsilon = \frac{R - H(\rho)}{2} > 0$. Then:

$$\operatorname{rk} \Pi_{n,\varepsilon} \overset{①}{\leq} 2^{n(H(\rho)+\varepsilon)} = 2^{n(R-\varepsilon)} \leq 2^{\lfloor nR \rfloor} = \dim (\mathbb{C}^2)^{\otimes \lfloor nR \rfloor} \quad \text{for } n \geq \frac{1}{\varepsilon}$$

$\Rightarrow \Pi_{n,\varepsilon} = V^\dagger V$ for some $V \in L(\mathcal{H}_A^{\otimes n}, (\mathbb{C}^2)^{\otimes \lfloor nR \rfloor})$

$$V = \sum_i |f_i\rangle\langle e_i|$$

Orthonormal vectors

basis of $S_{n,\varepsilon}$

Now define the channels

$$\mathcal{E}[M] = V M V^\dagger + \boxed{\operatorname{tr}[(I - V^\dagger V) M] \cdot \sigma}$$

$$\mathcal{D}[M] = V^\dagger M V + \boxed{\operatorname{tr}[(I - V V^\dagger) M] \cdot \omega}$$

arbitrary states

needed so that $\mathcal{E}, \mathcal{D}$ are trace-preserving

$\Longrightarrow$ $\mathcal{D}\circ\mathcal{E}$ has kraus operators $\{V^{+}V = \Pi_{n,\varepsilon}, \ldots\}$ and so

$$F(\mathcal{D}\circ\mathcal{E}, \rho^{\otimes n}) \geq \mathrm{tr}[\Pi_{n,\varepsilon}\rho^{\otimes n}] \xrightarrow{②} 1, \text{ hence } \geq 1-\delta \text{ for large } n \qquad \square$$

You proved part ② on the practice problems. We can discuss the following if you have any questions:

<u>Proof of Schumacher's theorem</u>, part ②: Crucial fact: If $P$ is orthogonal proj. of rank $\leq 2^{nR}$ then, for $\varepsilon = \frac{H(\rho)-R}{2} > 0$:

$$\mathrm{tr}[P\rho^{\otimes n}] = \underbrace{\mathrm{tr}[P\,\Pi_{n,\varepsilon}\rho^{\otimes n}]}_{\substack{\leq \|P\|_1 \cdot \|\Pi_{n,\varepsilon}\rho^{\otimes n}\Pi_{n,\varepsilon}\|_\infty \\ \leq 2^{nR}\, 2^{-n(H(\rho)-\varepsilon)} \leq 2^{-\varepsilon n}}} + \underbrace{\mathrm{tr}[P(I-\Pi_{n,\varepsilon})\rho^{\otimes n}]}_{\substack{\leq \|P\|_\infty \cdot \|(I-\Pi_{n,\varepsilon})\rho^{\otimes n}\|_1 \\ \leq 1-\mathrm{tr}[\Pi_{n,\varepsilon}\rho^{\otimes n}]}}$$

$$\leq 2^{-\varepsilon n} + (1-\mathrm{tr}[\Pi_{n,\varepsilon}\rho^{\otimes n}]) \longrightarrow 0 \quad \text{uniformly in } P$$

Now choose Kraus ops $\{X_i\}$ for $\mathcal{E}$, $\{Y_j\}$ for $\mathcal{D}$

$\rightsquigarrow$ kraus ops $\{Z_k\} = \{Y_j X_i\}$ for $\mathcal{D}\circ\mathcal{E}$ have rank $\leq 2^{nR}$

$\rightsquigarrow$ same is true for $P_k$ = orthogonal projection onto range of $Z_k$

Then:

$$F(\mathcal{D}\circ\mathcal{E}, \rho^{\otimes n})^2 = \sum_k |\mathrm{tr}[Z_k\rho^{\otimes n}]|^2 = \sum_k |\mathrm{tr}[P_k Z_k \rho^{\otimes n}]|^2$$

$$= \sum_k \underbrace{|\mathrm{tr}[Z_k\sqrt{\rho^{\otimes n}}\sqrt{\rho^{\otimes n}}P_k]|^2}_{\leq \|Z_k\sqrt{\rho^{\otimes n}}\|_2^2 \cdot \|\sqrt{\rho^{\otimes n}}P_k\|_2^2} \leq \sum_k \underbrace{\mathrm{tr}[Z_k^{+}Z_k\rho^{\otimes n}]}_{\text{probability dist!}}\,\mathrm{tr}[P_k\rho^{\otimes n}] \longrightarrow 0$$

$\square$

# Entropies of subsystems

**NOTATION:** $p_{XY} \in P(\Sigma_X \times \Sigma_Y) \rightsquigarrow p_X(x) = \sum_y p_{XY}(x,y)$, $p_Y(y) = \dots$

$$H(XY)_p := H(p_{XY}), \quad H(X) = H(p_X), \quad H(Y) = H(p_Y),$$

usually omitted

Likewise for q. states: $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B) \rightsquigarrow \rho_A = \mathrm{tr}_B[\rho_{AB}]$, $\rho_B \in \mathrm{tr}_A[\rho_{AB}]$

$$H(AB)_\rho := H(\rho_{AB}), \quad H(A) = H(\rho_A), \quad H(B) = H(\rho_B)$$

usually omitted

Similarly if more than two subsystems.

* If $\rho_{AB}$ pure: $H(AB) = 0$ & $\underbrace{H(A) = H(B)}$ ← "entanglement entropy"

Pf: Clear. $\rho_A$ $\rho_B$ have same nonzero eigenvalues thanks to Schmidt decomposition: $|\Psi_{AB}\rangle = \sum_i s_i |e_i\rangle \otimes |f_i\rangle$ □

* If $\rho_{AB} = \rho_A \otimes \rho_B$: $H(AB) = H(A) + H(B)$

NB: Notation consistent! Pf: If $\rho_A$ eigenvalues $\{p_i\}$ & $\rho_B$ $\{q_j\}$ → $\rho_{AB}$ $\{p_i q_j\}$ □

Properties:

* $\boxed{H(AB) \leq H(A) + H(B)}$ Subadditivity (SA)

* In general: $H(AB) \not\geq H(A)$ or $H(B)$ but true for prob. dists.

you proved this on $\boxed{\text{PSET 6.}}$

* $H(AB) \geq |H(A) - H(B)|$ Araki-Lieb inequality (AL) weaker than monotonicity!

Pf: $\rho_{AB} \to$ purification $|\Psi_{ABC}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

Useful trick!

$\Rightarrow H(AB) = H(C) \overset{SA}{\geq} H(AC) - H(A) = H(B) - H(A)$ □

* $\boxed{H(AC) + H(BC) \geq H(ABC) + H(C)}$ Strong subadditivity (SSA)

SSA ⇒ SA if no C

NONTRIVIAL! Can add proof to lecture notes if there is interest.

* equivalent: $H(AB) + H(BC) \geq H(A) + H(C)$ weak monotonicity (WM)

↳ $\boxed{\text{EX CLASS}}$

# Mutual information

**Def:** $I(A:B) = H(A) + H(B) - H(AB)$

e.g. what we "lose" when we treat A,B independently in compression

* defined for q. states $\rho_{AB}$ & probability distributions $p_{XY} \longrightarrow I(X:Y)$

relation: if $\rho_{XY} = \sum_{x,y} p(x,y)\, |x\rangle\langle x| \otimes |y\rangle\langle y|$: $\boxed{I(X:Y)_\rho = I(X:Y)_p}$

* e.g. $\rho_{AB} = |\Phi^+\rangle\langle\Phi^+|$, $|\Phi^+\rangle = \sqrt{\frac{1}{2}}\left(|00\rangle + |11\rangle\right)$: $\quad I(A:B) = 1+1-0 = 2$

$\rho_{XY} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$: $\quad I(X:Y) = 1+1-1 = 1$

## Properties:

* $\boxed{I(A:B) \overset{SA}{\geq} 0, \; = 0 \text{ iff } \rho_{AB} = \rho_A \otimes \rho_B}$

"only if"? proof next week!

quantifies correlations

* invariant under isometries $\mathcal{H}_A \to \mathcal{H}_{\tilde{A}}$ or $\mathcal{H}_B \to \mathcal{H}_{\tilde{B}}$

* $\rho_{AB}$ pure: $I(A:B) = 2H(A) = 2H(B)$

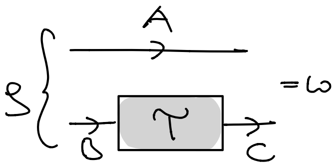
* $I(A:B) \overset{AL}{\leq} 2 \cdot \min\{H(A), H(B)\} \leq 2 \cdot \log \min\{d_A, d_B\}$ "="? $\rightarrow$ $\boxed{\text{PSET}}$

no factor 2 for prob. dist.
(use monotonicity)

* $I(A:C) \overset{SSA}{\leq} I(A:CE)$ for all $\rho_{ACE}$. More generally:

**Data Processing Inequality (DPI):** For all $\rho_{AB}$ & $T_{B\to C}$:

$$I(A:C)_\omega \leq I(A:B)_\rho \quad \text{where } \omega_{AC} = (I_A \otimes T_{B\to C})[\rho_{AB}]$$



Intuitive: CANNOT increase correlations by acting locally!

* generalizes SSA: $B = CE$, $T = \mathrm{tr}_E$.

* follows from SSA: choose Stinespring isometry $V_{B\to CE}$ for $T_{B\to C}$. Then:

$$\omega_{ACE} = (I_A \otimes V_{B\to CE})\, \rho_{AB}\, (I_A \otimes V^\dagger_{B\to CE}) \quad \text{extends } \omega_{AC}$$

$$\Rightarrow I(A:B)_\rho \underset{\substack{\text{invariance}\\ \text{under isometries}}}{=} I(A:CE)_\omega \underset{SSA}{\geq} I(A:B)_\omega$$

□