Quantum Information Theory, Spring 2020

Practice problem set #6

You do **not** have to hand in these exercises, they are for your practice only.

1. **Trace distance of probability distributions:** In today's lecture, we defined the *(normalized) trace distance* between $p, q \in P(\Sigma)$ by $T(p, q) := \frac{1}{2} \sum_{x \in \Sigma} |p(x) - q(x)|$. This quantity is also known as the *total variation distance* between p and q.

   (a) Show that $T(p, q) = T(\rho, \sigma)$, where $\rho = \sum_x p(x)|x\rangle\langle x|$ and $\sigma = \sum_x q(x)|x\rangle\langle x|$.

   (b) Let X, Y be random variables with distributions p, q, respectively. Show that

   $$T(p, q) = \max_{S \subseteq \Sigma} \big(\Pr(X \in S) - \Pr(Y \in S)\big).$$

   Do you recognize this as the probability theory analog of a formula that you proved for quantum states?

   (c) Suppose X, Y are random variables as above and have a joint distribution. Use part (b) to show that $T(p, q) \leqslant \Pr(X \neq Y)$. This beautiful inequality is known as the *coupling inequality*.

2. **Compression vs correlations:** In today's lecture we characterized (n, R, δ)-codes in terms of how well they preserve correlations with an auxiliary random variable. Revisit the proof sketch in light of the results from Problem 1 and make sure you understand each step.

3. **On the definition of quantum codes:** The definition of an (n, R, δ)-quantum code in the lecture was perhaps surprising. Why did we not simply demand that $F(\mathcal{D}[\mathcal{E}[\rho^{\otimes n}]], \rho^{\otimes n}) \geqslant 1 - \delta$? Argue that such a definition would not correspond to a reliable compression protocol. What is the probability theory analog of this condition?

4. **Converse of Schumacher's theorem:** In this problem you can try to prove part 2 of Schumacher's theorem yourself. Fix $\rho \in D(\mathcal{H}_A)$, $\delta \in (0, 1)$, and $R < H(\rho)$.

   (a) Show that there exists $\varepsilon > 0$ such that, for any orthogonal projection P of rank $\leqslant 2^{nR}$,

   $$\mathrm{Tr}[P\rho^{\otimes n}] \leqslant 2^{-\varepsilon n} + \big(1 - \mathrm{Tr}[\Pi_{n,\varepsilon}\rho^{\otimes n}]\big).$$

   (b) Show that, in any (n, R, δ)-code for ρ, the Kraus operators of $\mathcal{D} \circ \mathcal{E}$ have rank $\leqslant 2^{nR}$.

   (c) Show that there exist (n, R, δ)-codes for ρ for at most finitely many values of n.
   Hint: Use the formula for the channel fidelity from class and the Cauchy-Schwarz inequality for operators.