Quantum Information Theory, Spring 2020

Homework problem set #6 **due March 16, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(3 points)* **Compression and correlations:** Let $\rho = \sum_{x \in \Sigma} p(x) \rho_x$, where $p \in P(\Sigma)$ is a probability distribution and ρ_x a state for each $x \in \Sigma$. In class, we showed that if $\mathcal{E}$ and $\mathcal{D}$ are channels such that $F(\mathcal{D} \circ \mathcal{E}, \rho^{\otimes n}) \geqslant 1 - \delta$ then

$$\sum_{x^n} p(x_1) \cdots p(x_n) \, F\big(\mathcal{D}\big[\mathcal{E}[\rho_{x_1} \otimes \cdots \otimes \rho_{x_n}]\big], \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}\big) \geqslant 1 - \delta.$$

   Show that the converse is not necessarily true.

   Hint: There are even counterexamples for $n = 1$ and $\delta = 0$. Consider measure-and-prepare channels.

2. *(2 points)* **Non-monotonicity of the von Neumann entropy:** Given a quantum state ρ_{AB}, we write $H(AB)$ for its entropy and $H(A)$, $H(B)$ for the entropies of its reduced states.

   (a) Find a state ρ_{AB} such that $H(AB) > H(B)$.
   (b) Find a state ρ_{AB} such that $H(AB) < H(B)$.

   Thus, the von Neumann entropy does *not* satisfy the same monotonicity as the Shannon entropy.

3. *(3 points)* **Subadditivity of the von Neumann entropy:** Use Schumacher's theorem to show that, for all states ρ_{AB},

$$H(A) + H(B) \geqslant H(AB),$$

   where we use the same notation as in the previous problem. Thus, *subadditivity* still holds.

   Hint: You may use the following 'triangle inequality' for the fidelity (without proof): For any three states $\alpha, \beta, \gamma \in D(\mathcal{H})$, if $F(\alpha, \beta) \geqslant 1 - \delta$ and $F(\beta, \gamma) \geqslant 1 - \delta$ then $F(\alpha, \gamma) \geqslant 1 - 4\delta$.

4. *(4 points)* **Classical-quantum states and concavity:** Given a probability distribution $p \in P(\Sigma)$ and states $\rho_x \in D(\mathcal{H})$ for $x \in \Sigma$, we can consider the cq state $\rho_{XB} = \sum_{x \in \Sigma} p(x) \, |x\rangle\langle x| \otimes \rho_x$ in $D(\mathcal{H}_X \otimes \mathcal{H}_B)$, where $\mathcal{H}_X = \mathbb{C}^\Sigma$ and $\mathcal{H}_B = \mathcal{H}$.

   (a) Show that $H(XB) = H(p) + \sum_{x \in \Sigma} p(x) H(\rho_x)$.
   (b) Conclude that $H(XB) \geqslant H(X)$. When does equality hold?
   (c) Show that the von Neumann entropy is a concave function on $D(\mathcal{H})$.
   Hint: Evaluate the subadditivity inequality from Problem 3 for a classical-quantum state.