Quantum Information Theory, Spring 2020

Practice problem set #4

You do **not** have to hand in these exercises, they are for your practice only.

1. **Functionals:** Let $\lambda\colon L(\mathcal{H}) \to \mathbb{C}$ be a linear function.
   (a) Show that there exists a unique $X \in L(\mathcal{H})$ such that $\lambda[M] = \mathrm{Tr}[X^\dagger M]$ for all $M \in L(\mathcal{H})$.
   (b) How about if $\lambda[M] \geqslant 0$ for all $M \geqslant 0$?

2. **Depolarizing and dephasing channels:** The *completely depolarizing channel* on $L(\mathcal{H})$ is given by

$$\mathcal{D}[M] = \mathrm{Tr}[M]\frac{I}{d} \qquad \forall M \in L(\mathcal{H}),$$

   where $d = \dim \mathcal{H}$. For $\mathcal{H} = \mathbb{C}^\Sigma$, the *completely dephasing channel* is defined by

$$\Delta[M] = \sum_x \langle x|M|x\rangle\, |x\rangle\langle x| \qquad \forall M \in L(\mathcal{H}).$$

   (a) Compute the Choi operator of either channel.
   (b) What is the result of acting by either channel on half of a maximally entangled state?
   (c) For qubits, $\mathcal{H} = \mathbb{C}^2$, how does either channel act on Bloch vectors?

3. **Kraus and Stinespring:** Find Kraus and Stinespring representations for the following quantum channels:
   (a) *Trace:* $\Phi[M] = \mathrm{Tr}[M]$
   (b) *Add pure state:* $\Phi[M_A] = M_A \otimes |\phi\rangle\langle\phi|_B$ for a unit vector $|\phi_B\rangle \in \mathcal{H}_B$.
   (c) *Completely dephasing channel:* $\Delta[M] = \sum_x \langle x|M|x\rangle\, |x\rangle\langle x|$ (same as above).

4. **Kraus and Stinespring:** Given Kraus or Stinespring representations of two channels $\Phi_{A\to B}$ and $\Psi_{B\to C}$, explain how to obtain the same representation for $\Psi_{B\to C} \circ \Phi_{A\to B}$.

5. **Stinespring with unitaries:** Use the Stinespring representation to prove that any quantum channel $\Phi_{A\to B}$ can be written in the following form:

$$\Phi_{A\to B}[M_A] = \mathrm{Tr}_E\left[U_{AC\to BE}(M_A \otimes \sigma_C)U^\dagger_{AC\to BE}\right] \qquad \forall M_A,$$

   where $\mathcal{H}_C, \mathcal{H}_E$ are auxiliary Hilbert spaces, $\sigma_C \in D(\mathcal{H}_E)$ is a *pure* state, and $U_{AC\to BE}$ a *unitary*.

6. **Adjoint superoperator:** Recall that the Hilbert-Schmidt inner product on $L(\mathcal{H})$ is given by $\langle M, N\rangle_{HS} = \mathrm{Tr}[M^\dagger N]$. This allows us to define the *adjoint* of a superoperator $\Phi \in L(L(\mathcal{H}_A), L(\mathcal{H}_B))$. Explicitly, this is the superoperator $\Phi^\dagger \in L(L(\mathcal{H}_B), L(\mathcal{H}_A))$ such that

$$\langle M_A, \Phi^\dagger[N_B]\rangle_{HS} = \langle \Phi[M_A], N_B\rangle_{HS} \qquad \forall M_A, N_B.$$

   (a) Given a Kraus representation of Φ, explain to find one for $\Phi^\dagger$.
   (b) Show that Φ is completely positive if and only if $\Phi^\dagger$ is completely positive.
   (c) Show that Φ is trace-preserving if and only if $\Phi^\dagger$ is unital (i.e., $\Phi^\dagger[I_B] = I_A$).