Quantum Information Theory, Spring 2020

Homework problem set #4 **due March 2, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(2 points)* **Monotonicity of distance measures:** Use the Stinespring representation to deduce the following monotonicity properties. For all states ρ_A, σ_A and channels $\Phi_{A\to B}$,

$$T(\Phi_{A\to B}[\rho_A], \Phi_{A\to B}[\sigma_A]) \leqslant T(\rho_A, \sigma_A) \quad \text{and} \quad F(\Phi_{A\to B}[\rho_A], \Phi_{A\to B}[\sigma_A]) \geqslant F(\rho_A, \sigma_A).$$

2. *(3 points)* **Depolarizing channel:** Consider the following trace-preserving superoperator on $L(\mathcal{H})$, where $\dim \mathcal{H} = d$ and $\lambda \in \mathbb{R}$ is a parameter:

$$\mathcal{D}_\lambda[M] = \lambda\, M + (1-\lambda)\,\mathrm{Tr}[M]\frac{I}{d}$$

   (a) Compute the Choi operator of $\mathcal{D}_\lambda$ for any value of λ.
   (b) For which values of λ is $\mathcal{D}_\lambda$ a quantum channel?

3. *(5 points)* **Kraus and Stinespring:** Find Kraus and Stinespring representations for the following quantum channels:

   (a) *Partial trace:* $\Phi[M_{AE}] = \mathrm{Tr}_E[M_{AE}]$
   (b) *Add state:* $\Phi[M_A] = M_A \otimes \sigma_B$ for a state σ_B.
   (c) *Measure and prepare:* $\Phi[M] = \sum_{x\in\Sigma} \langle x|M_A|x\rangle\, \sigma_{B,x}$, where $|x\rangle$ denotes the standard basis of $\mathcal{H}_A = \mathbb{C}^\Sigma$ and $\sigma_{B,x}$ is an arbitrary state for each $x \in \Sigma$.

4. *(2 points)* **Quantum to classical channels:** Let $\mathcal{H}_A$ be an arbitrary Hilbert space and $\mathcal{H}_X = \mathbb{C}^\Omega$. Assume that $\Phi_{A\to X}$ is a quantum channel such that $\Phi_{A\to X}[\rho_A]$ is classical for every state ρ_A. Show that there exists a measurement $\mu_A : \Omega \to \mathrm{PSD}(\mathcal{H}_A)$ such that

$$\Phi_{A\to X}[\rho_A] = \sum_{x\in\Omega} \mathrm{Tr}\big[\mu_A(x)\rho_A\big]\, |x\rangle\langle x| \qquad \forall \rho_A.$$

   Hint: Use Practice Problem 4.1.