Structure of Quantum Channels

Last week: $\Phi_{A\to B}$ is "superoperator" that is completely positive & trace-preserving

$L(\mathcal{H}_A) \to L(\mathcal{H}_B)$ $\quad$ $\Phi \otimes \mathcal{I}_R$ positive $(\forall R)$ $\quad$ $\operatorname{tr} \circ \Phi = \operatorname{tr}$

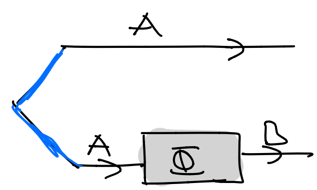
$\boxed{\text{Today:}}$ How to determine if a superoperator is a q. channel? Normal forms?

$\boxed{\text{Key tool:}}$ Choi operator of superoperator $\Phi_{A\to B}$:

fixed basis of $\mathcal{H}_A$

$$J^{\Phi}_{AB} = \sum_{x,y} |x\rangle\langle y| \otimes \Phi\big[|x\rangle\langle y|\big] \quad \in \operatorname{Lin}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

$$= (\mathcal{I}_A \otimes \Phi_{A\to B})\Big[\underbrace{\sum_{x,y} |x,x\rangle\langle y,y|}_{\text{(unnormalized) maximally entangled state}}\Big]$$



* e.g., Completely dephasing channel $\Delta[\rho] = \sum_z \langle z|\rho|z\rangle\, |z\rangle\langle z|$ on $\mathbb{C}^Z$

$$\Rightarrow J^{\Delta} = \sum_{x,y} |x\rangle\langle y| \otimes \underbrace{\Delta\big[|x\rangle\langle y|\big]}_{=\delta_{xy}|x\rangle\langle x|} = \sum_x |x\rangle\langle x| \otimes |x\rangle\langle x|$$

* Choi-Jamiołkowski isomorphism: $L(L(\mathcal{H}_A), L(\mathcal{H}_B)) \xrightarrow{\cong} L(\mathcal{H}_A \otimes \mathcal{H}_B)$

$$\Phi \mapsto J^{\Phi}$$

Inverse?

$$\boxed{\Phi_{A\to B}[M_A] = \operatorname{tr}_A\big[(M_A^T \otimes I_B)\, J^{\Phi}_{AB}\big]}$$

Pf: $\text{RHS} = \sum_{x,y} \operatorname{tr}_A\big[(M_A^T |x\rangle\langle y|) \otimes \Phi[|x\rangle\langle y|]\big] = \sum_{x,y} \underbrace{\operatorname{tr}\big[M_A^T |x\rangle\langle y|\big]}_{=\langle y|M_A^T|x\rangle = \langle x|M_A|y\rangle} \Phi[|x\rangle\langle y|]$

$$= \sum_{x,y} \Phi\big[|x\rangle\langle x|M_A|y\rangle\langle y|\big] = \Phi[M_A].$$

$\square$

In the example: $\operatorname{tr}\big[(M_A^T \otimes I_B)\, J^{\Delta}_{AB}\big] = \sum_x \langle x|M_A^T|x\rangle\, |x\rangle\langle x| = \Delta[\rho]$ ✓

Complete Positivity

Theorem: For a superoperator $\Phi_{A\to B} \in L(L(\mathcal{H}_A), L(\mathcal{H}_B))$, the followings are equi:

1. $\Phi_{A\to B}$ is completely positive $\leftarrow \forall \mathcal{H}_R$: $\Phi\otimes\mathcal{I}_R$ is positive
2. $\Phi_{A\to B}\otimes\mathcal{I}_A$ is positive !
3. $J^{\Phi}_{AB}\geq 0$!
4. Kraus representation: $\exists X_1,\dots,X_r \in L(\mathcal{H}_A,\mathcal{H}_B)$: $\Phi[M]=\sum_i X_i M X_i^\dagger \quad \forall M$ ("kraus operators")
5. Stinespring representation: $\exists \mathcal{H}_E$, $V\in L(\mathcal{H}_A,\mathcal{H}_B\otimes\mathcal{H}_E)$: $\Phi[M]=\mathrm{tr}_E[VMV^\dagger]\ \forall M$ ("Stinespring extension")

Moreover: In (4) and (5), r and $\dim \mathcal{H}_E$ can be chosen $= \operatorname{rank} J^{\Phi}_{AB} \leq d_A d_B$.

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) ✓

(5) $\Rightarrow$ (1) since both $M\mapsto VMV^\dagger$ and tr_E are completely positive ✓ cf. $\boxed{\text{EX 3.3(a)}}$

(4) $\Rightarrow$ (5): $\mathcal{H}_E := \mathbb{C}^r$ & $V_{A\to BE} := \sum_{i=1}^r X_i\otimes|i\rangle$ ✓

$\Rightarrow \mathrm{tr}_E[VMV^\dagger] = \sum_{i,j} X_i M X_j^\dagger \cdot \mathrm{tr}[|i\rangle\langle j|] = \sum_i X_i M X_i^\dagger$

(4) $\Leftarrow$ (5)? $X_i := (\mathcal{I}_B\otimes\langle i|)V$

(3) $\Rightarrow$ (4): Using spectral decomposition, write

$$J^{\Phi}_{AB} = \sum_i |v_i\rangle\langle v_i| \quad \text{for } |v_i\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B \quad \leftarrow \|v_i\|^2 = \text{eigenvalues !}$$

Define $X_i = \sum_{a,b} \langle ab|v_i\rangle\, |b\rangle\langle a| \in L(\mathcal{H}_A,\mathcal{H}_B)$ Turn into operator!

$$\Rightarrow \Phi[M] = \sum_i \mathrm{tr}_A[(M_A^T\otimes I_B)|v_i\rangle\langle v_i|]$$

$$= \sum_i \sum_{a,b}\sum_{\tilde a,\tilde b} \langle ab|v_i\rangle\langle v_i|\tilde a\tilde b\rangle\, \mathrm{tr}_A[(M_A^T\otimes I_B)\,|ab\rangle\langle\tilde a\tilde b|]$$

no more transpose

$$= \sum_i \sum_{a,b}\sum_{\tilde a,\tilde b} \langle ab|v_i\rangle\langle v_i|\tilde a\tilde b\rangle\, \langle a|M_A|\tilde a\rangle\, |b\rangle\langle\tilde b|$$

$$= \sum_i X_i M_A X_i^\dagger$$

$\square$

Ex: $\Delta[\rho] = \sum_z \langle z|\rho|z\rangle \, |z\rangle\langle z|$?

Kraus: $X_z = |z\rangle\langle z|$ for $z \in \Sigma$

Stinespring: $\mathcal{H}_E = \mathcal{H}_A$, $V = \sum_z |zz\rangle\langle z| = \sum_z |z\rangle\langle z| \otimes |z\rangle$

implies that completely positive

Quantum Channels

When is a completely positive superoperator trace-preserving?

* Choi: $\mathrm{tr}_B[J^\Phi_{AB}] = I_A$
* Kraus: $\sum_i X_i^\dagger X_i = I_A$ for one/any Kraus representation
* Stinespring: $\underbrace{V^\dagger V = I_A}_{\text{i.e. } V \text{ isometry}}$ for one/any Stinespring extension

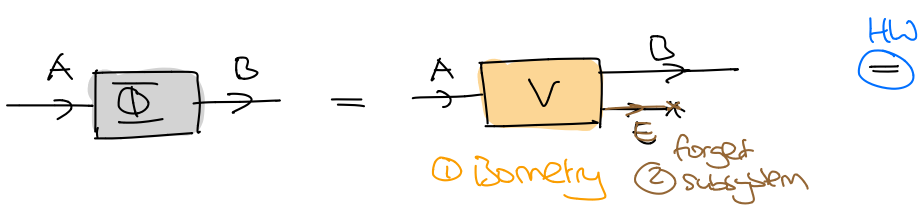
Pf: Kraus: $\mathrm{tr}[M_A I_A] = \mathrm{tr}[M_A] \overset{!}{=} \mathrm{tr}[\sum_i X_i M_A X_i^\dagger] = \mathrm{tr}[M_A \sum_i X_i^\dagger X_i]$ $\forall M_A$

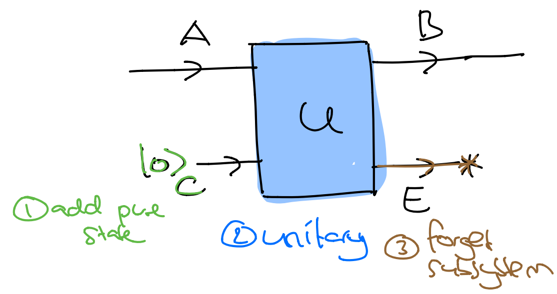
Stinespring: $\mathrm{tr}[M_A I_A] = \mathrm{tr}[M_A] \overset{!}{=} \mathrm{tr}[V M_A V^\dagger] = \mathrm{tr}[M_A V^\dagger V]$ $\forall M_A$

Choi: Similar. $\square$

Aside: $\frac{1}{d_A} J^\Phi_{AB} = (\mathcal{I}_A \otimes \Phi_{A\to B})[|\Phi^+\rangle\langle\Phi^+|]$ is called Choi state (⇝ "state-channel duality")

$\boxed{\text{NB:}}$ Stinespring representation for q. channels





⇝ GOOD DEFINITION 😀