Quantum Information Theory, Spring 2020

Homework problem set #3 **due February 24, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(3 points)* **Pretty good measurement:** Let $\rho_1, \dots, \rho_n$ be a collection of quantum states on some Hilbert space $\mathcal{H}$ with the property that $I \in \operatorname{span}\{\rho_1, \dots, \rho_n\}$.

   (a) Show that $Q := \sum_{i=1}^n \rho_i$ is positive semidefinite and invertible. *Hint: For the latter, assume* Q *has an eigenvector* $|\psi\rangle$ *with eigenvalue* 0*. Use the span property to get a contradiction.*
   (b) Define $\mu\colon \{1, \dots, n\} \to L(\mathcal{H})$ by $\mu(i) = Q^{-1/2} \rho_i Q^{-1/2}$. Show that μ is a measurement.

2. *(3 points)* **Properties of the fidelity:** Use Uhlmann's theorem to prove the following properties of the fidelity. As always, $\mathcal{H}_A$ and $\mathcal{H}_B$ denote arbitrary Hilbert spaces.

   (a) *Monotonicity:* $F(\rho_{AB}, \sigma_{AB}) \leqslant F(\rho_A, \sigma_A)$ for any two states $\rho_{AB}, \sigma_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$.
   (b) *Joint concavity:* $\sum_{i=1}^n p_i F(\rho_i, \sigma_i) \leqslant F(\sum_{i=1}^n p_i \rho_i, \sum_{i=1}^n p_i \sigma_i)$, where $(p_i)_{i=1}^n$ is an arbitrary probability distribution and $\rho_1, \dots, \rho_n$ and $\sigma_1, \dots, \sigma_n$ are states in $D(\mathcal{H}_A)$.

   Hint: For both (a) and (b), try to find suitable purifications.

3. *(3 points)* **Quantum channels:** Show that the following maps Φ are quantum channels by verifying that they are completely positive and trace-preserving:

   (a) *Mixture of unitaries:* $\Phi[M] = \sum_{i=1}^n p_i U_i M U_i^\dagger$, where $(p_i)_{i=1}^n$ is an arbitrary probability distribution and $U_1, \dots, U_n$ arbitrary unitaries.
   (b) *State replacement:* $\Phi[M] = \operatorname{Tr}[M]\, \sigma$, where σ is an arbitrary state.
   (c) *Measure and prepare:* $\Phi[M] = \sum_{x \in \Sigma} \langle x|M|x\rangle\, \sigma_x$, where $|x\rangle$ denotes the standard basis of $\mathbb{C}^\Sigma$ and σ_x is an arbitrary state for each $x \in \Sigma$.

4. *(3 points)* **No cloning:** In this problem, you will show that it is not possible to perfectly clone an unknown state – even if we restrict to classical or to pure states. Let $\mathcal{H} = \mathbb{C}^2$ be a qubit. We say that a channel $\Phi \in C(\mathcal{H}, \mathcal{H} \otimes \mathcal{H})$ *clones* a state $\rho \in D(\mathbb{C}^2)$ if $\Phi[\rho] = \rho \otimes \rho$.

   (a) Show that there exists no channel that clones all classical states ρ.
   (b) Show that there exists no channel that clones all pure states ρ.
   (c) Which states are both pure *and* classical? Find a channel Φ that clones all of them.

   Hint: For (a) and (b), use that channels are linear to arrive at a contradiction.

5. *(2 bonus points)* 🖩 **Practice:** Here you can verify that the measurement from Problem 1 is 'pretty good' at distinguishing the states $\rho = |0\rangle\langle 0|$ and $\sigma(t) = (1-t)|0\rangle\langle 0| + t|1\rangle\langle 1|$. Assuming both states are equally likely, plot the following two quantities as functions of $t \in (0, 1]$:

   (a) The optimal probability of distinguishing ρ and $\sigma(t)$. *Hint: Helstrom's theorem.*
   (b) The probability of distinguishing ρ and $\sigma(t)$ by using the measurement from Problem 1.