Quantum Information Theory, Spring 2020

Practice problem set #2

You do **not** have to hand in these exercises, they are for your practice only.

Throughout, A, B, C denote quantum systems with Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$. The sets $\{|a\rangle\}$ and $\{|b\rangle\}$ denote arbitrary orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$; $|a,b\rangle = |a\rangle \otimes |b\rangle$ denotes the product basis.

1. **Singular values and eigenvalues:** Recall that the *singular values* of an operator M are the square roots of the nonzero eigenvalues of $M^\dagger M$ or $MM^\dagger$ (which are always positive semidefinite).

   (a) Show that if M is Hermitian then its singular values are equal to its nonzero *absolute* eigenvalues. How about if the operator is positive semidefinite?
   (b) Argue that $\|M\|_1 = \mathrm{Tr}[M]$ if M is positive semidefinite. In particular, $\|\rho\|_1 = 1$ for any state.

2. **Reduced states of classical states:** Consider the 'classical' state $\rho_{XY} = \sum_{x,y} p(x,y)\,|x,y\rangle\langle x,y|$ on $\mathcal{H}_X \otimes \mathcal{H}_Y$, where $\mathcal{H}_X = \mathbb{C}^{\Sigma_X}$, $\mathcal{H}_Y = \mathbb{C}^{\Sigma_Y}$, and $p(x,y)$ is an arbitrary probability distribution. Compute the reduced states ρ_X and ρ_Y.

3. **Reduced states of a pure state:** Compute the reduced states ρ_A and ρ_B of the two-qubit pure state $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ given by $|\Psi_{AB}\rangle = \frac{1}{3}|0,0\rangle + \frac{2}{3}|0,1\rangle + \frac{2}{3}|1,0\rangle$.

   Hint: If this calculation seems too painful to carry out, see the next problem.

4. **Schmidt decomposition:** Let $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ be an arbitrary pure state. Fix bases $|a\rangle$ and $|b\rangle$, write $|\Psi_{AB}\rangle = \sum_{a,b} M_{ab}|a,b\rangle$, and define a corresponding operator $M = \sum_{a,b} M_{ab}|a\rangle\langle b|$.

   (a) Verify that $\rho_A = MM^\dagger$ and $\rho_B = M^T\overline{M}$.
   (b) Let $M = \sum_i s_i|e_i\rangle\langle f_i|$ be a singular value decomposition. Show that $|\Psi_{AB}\rangle = \sum_i s_i|e_i\rangle \otimes |\overline{f_i}\rangle$ is a Schmidt decomposition.
   (c) Explain how to find a Schmidt decomposition of the following two-qubit pure state:

   $$|\Psi\rangle = \frac{\sqrt{2}+1}{\sqrt{12}}\Big(|00\rangle + |11\rangle\Big) + \frac{\sqrt{2}-1}{\sqrt{12}}\Big(|01\rangle + |10\rangle\Big)$$

   Note: The transpose and the complex conjugate are computed with respect to the fixed bases.

5. **Partial trace trickery:** Verify the following calculational rules for the partial trace:

   (a) Let $X_A, Y_A \in L(\mathcal{H}_A)$ and $M_{AB} \in L(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then, $\mathrm{Tr}_B[(X_A \otimes I_B)M_{AB}(Y_A \otimes I_B)] = X_A \,\mathrm{Tr}_B[M_{AB}]Y_A$.
   (b) Let $N_{ABC} \in L(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Then, $\mathrm{Tr}_{AB}[N_{ABC}] = \mathrm{Tr}_A[\mathrm{Tr}_B[N_{ABC}]] = \mathrm{Tr}_B[\mathrm{Tr}_A[N_{ABC}]]$.

   What does this last rule look like if there is no C-system?

6. **Observables (for those of you who have taken a quantum mechanics course):** In this problem we discuss the relationship between 'measurements' as defined in the lecture and 'observables' as introduced in a first quantum mechanics course. An *observable* on a quantum system is by definition a Hermitian operator on the corresponding Hilbert space $\mathcal{H}$.

(a) Let $\mu\colon \Omega \to \mathrm{PSD}(\mathcal{H})$ be a *projective* measurement with outcomes in the real numbers, i.e., a finite subset $\Omega \subseteq \mathbb{R}$. Show that the following operator is an observable:

$$\mathcal{O} = \sum_{\omega \in \Omega} \omega \, \mu(\omega) \tag{1}$$

In fact, this is always an eigendecomposition, but you need not prove this.

(b) Argue that, conversely, any observable can be written as in Eq. (1) for some suitable μ.

(c) Now suppose that the system is in state ρ and we perform the measurement μ. Show that the *expectation value* of the measurement outcome is given by $\mathrm{Tr}[\rho\mathcal{O}]$.
For a pure state $\rho = |\psi\rangle\langle\psi|$, this can also be written as $\langle\psi|\mathcal{O}|\psi\rangle$. Do you recognize these formulas from your quantum mechanics class?