Joint Systems, Reduced States, Purifications

$\boxed{\text{Last week:}}$ States + Measurements

"density operators"
$\rho \in PSD(\mathcal{H}), \ tr[\rho]=1$

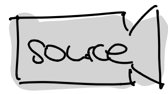
"POVM"
$\mu : \Omega \to PSD(\mathcal{H}), \ \sum_\omega \mu(\omega) = I$

$\Big\}$ $Pr(\text{outcome } \omega) = tr[\rho\, \mu(\omega)]$

States are pure if $\rho = |\psi\rangle\langle\psi|$. Otherwise: mixed. How do they arise?

* classical states: $\rho = \sum_{x\in\Sigma} p(x)\, |x\rangle\langle x|$ if $\mathcal{H} = \mathbb{C}^\Sigma$
  prob. dist.

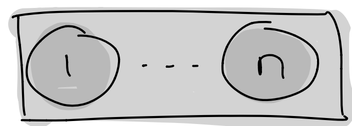
* ensembles: $\{p_i, \rho_i\}_{i\in I} \rightsquigarrow$ average state $\rho = \sum_i p_i \rho_i$



or ensembles in statistical physics

$\boxed{\text{Today:}}$ Also arise when describing the state of subsystems!

$\boxed{\text{For a (q.) system composed of subsystems the Hilbert space is } \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n.}$



e.g. n qubits: $\mathcal{H} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 = (\mathbb{C}^2)^{\otimes n}$

e.g. $\mathcal{H} = \mathcal{H}^{\Sigma_1} \otimes \cdots \otimes \mathcal{H}^{\Sigma_n} \cong \mathcal{H}^{\Sigma}$ for $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ $\quad \Sigma_i = \{0,1\}$

w/ product basis $|x_1,\dots,x_n\rangle := |x_1\rangle \otimes \cdots \otimes |x_n\rangle$ $\quad (x_i \in \Sigma_i)$

often leave out commas $\qquad$ tensor product of vectors

States?

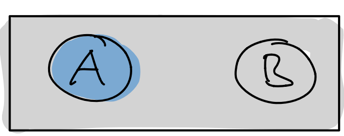
* product states: $\rho = \rho_1 \otimes \cdots \otimes \rho_n$ where $\rho_i \in D(\mathcal{H}_i)$ $\qquad \hat{=}$ independent random variables
  tensor product of operators

* otherwise: correlated $\qquad$ e.g. $\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| = \begin{pmatrix} \frac{1}{2} & & & \\ & 0 & & \\ & & 0 & \\ & & & \frac{1}{2} \end{pmatrix}$
  $\in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$

$\boxed{\text{NOTATION:}}$ From now on, we use subscripts to indicate (sub)systems.

E.g. $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, ρ_{AB} for state on $\mathcal{H}_{AB}$, X_B for operator on $\mathcal{H}_B$, ...

$\boxed{\text{Question:}}$ Given state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$,
how to describe state on subsystem A?



Clear if $\rho_{AB} = \rho_A \otimes \rho_B$ — but in general? Need input from q. theory:

$$\boxed{\begin{array}{l}\text{To measure } \mu_A : \Omega \to PSD(\mathcal{H}_A) \text{ on subsystem: Use} \\ \quad \mu_A \otimes I_B : \Omega \to PSD(\mathcal{H}_A \otimes \mathcal{H}_B), \ \omega \mapsto \mu_A(\omega) \otimes I_B\end{array}}$$

measurement on AB

If we measure in state ρ_{AB}:

$$\Pr(\text{outcome } \omega) = \operatorname{tr}[\rho_{AB}(\mu_A(\omega) \otimes I_B)]$$

evaluate trace in $\otimes$ orthonormal basis

$$= \sum_{a,b} (\langle a| \otimes \langle b|) \rho_{AB} (\mu_A(\omega) \otimes I_B) (|a\rangle \otimes |b\rangle)$$

$$= \sum_{a,b} \langle a| (I_A \otimes \langle b|) \rho_{AB} (\mu_A(\omega) \otimes I_B)(I_A \otimes |b\rangle) |a\rangle$$

commute

$$= \sum_a \langle a| \sum_b (I_A \otimes \langle b|) \rho_{AB} (I_A \otimes |b\rangle) \mu_A(\omega) |a\rangle$$

$$= \operatorname{tr}\Big[\underbrace{\sum_b (I_A \otimes \langle b|) \rho_{AB} (I_A \otimes |b\rangle)}_{\text{operator on } \mathcal{H}_A!} \mu_A(\omega) \Big]$$

Let's turn this into a def'n:

$$\boxed{\begin{array}{l}\text{The partial trace } \operatorname{tr}_B : L(\mathcal{H}_A \otimes \mathcal{H}_B) \to L(\mathcal{H}_A) \text{ is defined by} \\ \quad \operatorname{tr}_B[M_{AB}] := \sum_b (I_A \otimes \langle b|) M_{AB} (I_A \otimes |b\rangle) \quad \text{any ONB of } \mathcal{H}_B\end{array}}$$

Concretely: $\langle a| \operatorname{tr}_B[M_{AB}] |\tilde{a}\rangle = \sum_b \langle a,b| M_{AB} |\tilde{a}, b\rangle$

Thus, if we measure μ_A in state ρ_{AB}:

$$\Pr(\text{outcome } \omega) = \operatorname{tr}[\rho_A \mu_A(\omega)] \quad \text{where } \rho_A = \operatorname{tr}_B[\rho_{AB}] \quad \text{"reduced state"}$$

↳ ρ_A is good description of state of subsystem A

Rules: ① $\operatorname{tr}_B[X_A \otimes Y_B] = X_A \cdot \operatorname{tr}[Y_B]$ "partial trace" 😊

② $\operatorname{tr}[M_{AB}(X_A \otimes I_B)] = \operatorname{tr}[\operatorname{tr}_B[M_{AB}] X_A]$ ← Same proof as above

↳ ρ_{AB} state $\Rightarrow$ $\rho_A = \operatorname{tr}_B[\rho_{AB}]$ state — $X_A = I_A$ shows $\operatorname{tr} = 1$; $X_A \geq 0$ shows PSD

③ $\operatorname{tr}_B[(X_A \otimes I_B) M_{AB}] = X_A \operatorname{tr}_B[M_{AB}]$

EX/HW

Example: $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$, $|\Phi^+_{AB}\rangle = \sqrt{\frac{1}{2}}\left(|0,0\rangle + |1,1\rangle\right)$ max. entangled state

$$\Rightarrow \rho_{AB} = \frac{1}{2}\Big(\underbrace{|0,0\rangle\langle 0,0|}_{=|0\rangle\langle 0|\otimes|0\rangle\langle 0|} + \underbrace{|0,0\rangle\langle 1,1|}_{=|0\rangle\langle 1|\otimes|0\rangle\langle 1|} + |1,1\rangle\langle 0,0| + |1,1\rangle\langle 1,1|\Big)$$

$$\Rightarrow \rho_A = \mathrm{tr}_B[\rho_{AB}] = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \begin{pmatrix} 1/2 & \\ & 1/2 \end{pmatrix}$$

NB: Even if ρ_{AB} pure, ρ_A can be mixed. Conversely:

> Any $\sigma_A \in D(\mathcal{H}_A)$ has a **purification**: $\exists \mathcal{H}_B$, $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ s.th.
>
> $$\mathrm{tr}_B\left[|\Psi_{AB}\rangle\langle\Psi_{AB}|\right] = \sigma_A$$
>
> mixed states $\triangleq$ subsystems of pure states

* $\mathrm{rk}(\sigma_A) \leq \dim \mathcal{H}_B$ necessary + sufficient   CHURCH OF LARGER H.SPACE!

* existence:

$$\sigma_A = \sum_{i=1}^{r} p_i |\psi_i\rangle\langle\psi_i| \quad \Rightarrow \quad |\Psi_{AB}\rangle = \sum_i \sqrt{p_i}\, |\psi_i\rangle \otimes |i\rangle$$

spectral decomposition, $r := \mathrm{rk}(\sigma_A)$ purification, $\mathcal{H}_B = \mathbb{C}^r$ any ONB

[Pf: as above]

* alternative: Standard purification w/ $\mathcal{H}_B = \mathcal{H}_A$

$$|\Psi^{std}_{AB}\rangle = \left(\sqrt{\sigma_A} \otimes I_B\right) \sum_x |x\rangle \otimes |x\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$$

square root of Herm. op.   any pair of ONB

Pf: $\mathrm{tr}_B\left[|\Psi_{AB}\rangle\langle\Psi_{AB}|\right] = \sum_{x,y} \sqrt{\sigma_A}\, \underbrace{\mathrm{tr}_B\left[|x,x\rangle\langle y,y|\right]}_{=|x\rangle\langle y|\,\delta_{xy}} \sqrt{\sigma_A} = \sqrt{\sigma_A} \sum_x |x\rangle\langle x| \sqrt{\sigma_A} = \sigma_A$ □

Uniqueness? $|\Psi_{AB}\rangle, |\tilde{\Psi}_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ purifications

$$\Rightarrow |\tilde{\Psi}_{AB}\rangle = (I_A \otimes U_B)|\Psi_{AB}\rangle \text{ for unitary } U_B \text{ on } \mathcal{H}_B$$

$U_B^\dagger U_B = U_B U_B^\dagger = I$

More generally: If $|\Psi_{AB}\rangle, |\tilde{\Psi}_{AC}\rangle$ purifications, $\dim \mathcal{H}_B \leq \dim \mathcal{H}_C$

$$\Rightarrow |\tilde{\Psi}_{AC}\rangle = (I_A \otimes V)|\Psi_{AB}\rangle \text{ for isometry } V: \mathcal{H}_B \to \mathcal{H}_C$$

$V^\dagger V = I_B$ → lecture notes for details

This follows readily from the following important result:

Schmidt decomposition: Any $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as:

$$|\Psi_{AB}\rangle = \sum_i s_i \, |e_i\rangle \otimes |f_i\rangle$$

$s_i > 0$; $|e_i\rangle$ orthonormal in $\mathcal{H}_A$; $|f_i\rangle$ orthonormal in $\mathcal{H}_B$

reduced states

$$\rho_A = \sum_i s_i^2 |e_i\rangle\langle e_i| \qquad \rho_B = \sum_i s_i^2 |f_i\rangle\langle f_i|$$

eigenvalues are the same

$\rightarrow$ $\boxed{\text{HW}}$

* For $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ pure: ρ_{AB} product $\Leftrightarrow$ ρ_A pure $\Leftrightarrow$ ρ_B pure
* Existence? Follows from...

Singular value decomposition: For all $M \in L(\mathcal{H}, K)$ there exist $s_i > 0$ and orthonormal $|e_i\rangle \in K$, $|g_i\rangle \in \mathcal{H}$ s.t.

$$M = \sum_i s_i |e_i\rangle\langle g_i|$$

$\rightarrow$ $\boxed{\text{EX CLASS}}$

* How to find? $\{s_i\} = \{$square root of nonzero eigenvalues of $M^\dagger M\}$ (or $MM^\dagger$)

  $|g_i\rangle :=$ corresponding orthonormal eigenvectors, $|e_i\rangle = \dfrac{M|g_i\rangle}{s_i}$ (or the other way)

* If $M = M^\dagger$: $\{s_i\} = \{$absolute nonzero eigenvalues of $M\}$, $|g_i\rangle = \pm |e_i\rangle$

Next week, we will discuss distance measures between q. states. As preparation:

Norms of operators: For arbitrary $M \in L(\mathcal{H}, K)$ w/ singular values $\{s_i\}$

* trace norm: $\|M\|_1 = \sum_i s_i$
* Frobenius norm: $\|M\|_2 = \left(\sum_i s_i^2\right)^{1/2}$
* operator norm: $\|M\|_\infty = \max_i s_i$

If $M = M^\dagger$: $\{s_i\} \rightsquigarrow \{$abs. eigenvalues$\}$

e.g. $T(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1$ trace distance of q. states $\rho, \sigma \in D(\mathcal{H})$