Quantum Information Theory, Spring 2020

Practice problem set #1

You do **not** have to hand in these exercises, they are for your practice only.

1. **Dirac notation quiz:** In the Dirac notation, every vector is written as a 'ket' $|\psi\rangle$ and every linear functional is written as a 'bra' $\langle\psi| = |\psi\rangle^\dagger$, where † denotes the adjoint. One can think of kets as column vectors and bras as row vectors. Hence, if $|\psi\rangle$ is a column vector, then $\langle\psi|$ denotes the row vector obtained by taking the *conjugate transpose* of the column vector.

   (a) Let $|\psi\rangle$ and $|\phi\rangle$ be vectors in $\mathbb{C}^n$ and A an $n \times n$ matrix. Which of the following expressions are syntactically correct? For those that do, what kind of object do they represent (e.g., numbers, vectors, ...)? Can you write them using 'ordinary' notation?

   i. $|\psi\rangle + \langle\phi|$
   ii. $|\psi\rangle\langle\phi|$
   iii. $A\langle\psi|$
   iv. $\langle\psi|A$
   v. $\langle\psi|A + \langle\psi|$
   vi. $|\psi\rangle\langle\phi| + A$
   vii. $|\psi\rangle\langle\phi|A$
   viii. $|\psi\rangle A\langle\phi|$
   ix. $\langle\psi|A|\phi\rangle$
   x. $\langle\psi|A|\phi\rangle + \langle\psi|\phi\rangle$
   xi. $\langle\psi|\phi\rangle\langle\psi|$
   xii. $\langle\psi|\phi\rangle A$

   (b) Let $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ be two pure states on the same system. Verify that

   $$\mathrm{Tr}[\rho\sigma] = |\langle\psi|\phi\rangle|^2.$$

   Hint: You may use that the trace is cyclic, i.e. $\mathrm{Tr}[ABC] = \mathrm{Tr}[CAB] = \mathrm{Tr}[BCA]$.

2. **Positive semidefinite operators:** Recall from class that an operator $A \in L(\mathcal{H})$ is called *positive semidefinite* if it is Hermitian and all its eigenvalues are nonnegative. We denote by $\mathrm{PSD}(\mathcal{H})$ the set of positive semidefinite operators on a Hilbert space $\mathcal{H}$. Argue that the following conditions are equivalent:

   (a) A is positive semidefinite.
   (b) $A = B^\dagger B$ for an operator $B \in L(\mathcal{H})$.
   (c) $A = B^\dagger B$ for an operator $B \in L(\mathcal{H}, \mathcal{K})$ and some Hilbert space $\mathcal{K}$.
   (d) $\langle\psi|A|\psi\rangle \geqslant 0$ for every $\psi \in \mathcal{H}$.
   (e) $\mathrm{Tr}[AC] \geqslant 0$ for every $C \in \mathrm{PSD}(\mathcal{H})$.

3. **Convexity:** Recall that a set S is *convex* if $px + (1-p)y \in S$ for every $x, y \in S$ and $p \in [0, 1]$.

   (a) Show that $\mathrm{PSD}(\mathcal{H})$ is convex and closed under multiplication by $\mathbb{R}_{\geqslant 0}$ (i.e., a *convex cone*).
   (b) Show that $D(\mathcal{H})$ is convex.

4. **Positive semidefinite order:** Given two operators A and B, we write $A \leqslant B$ if the operator $B - A$ is positive semidefinite. Show that the following three conditions are equivalent:

   (a) $0 \leqslant A \leqslant I$.
   (b) A is Hermitian and has eigenvalues in $[0, 1]$.
   (c) $\langle\psi|A|\psi\rangle \in [0, 1]$ for every *unit vector* $|\psi\rangle \in \mathcal{H}$.

5. **Bloch sphere:** Recall from the lecture that the state ρ of a single qubit can be parameterized by the Bloch vector $\vec{r} \in \mathbb{R}^3$, $\|\vec{r}\| \leqslant 1$. Namely:

$$\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z).$$

(a) Show that $r_x = \mathrm{Tr}[\rho X]$, $r_y = \mathrm{Tr}[\rho Y]$, and $r_z = \mathrm{Tr}[\rho Z]$.

(b) Let σ be another qubit state, with Bloch vector $\vec{s}$. Verify that $\mathrm{Tr}[\rho\sigma] = \frac{1}{2}\left(1 + \vec{r} \cdot \vec{s}\right)$.

(c) Let $\{|\psi_i\rangle\}_{i=0,1}$ denote an orthonormal basis of $\mathbb{C}^2$, $\mu\colon \{0,1\} \to \mathrm{PSD}(\mathbb{C}^2)$ the corresponding basis measurement (i.e., $\mu(i) = |\psi_i\rangle\langle\psi_i|$ for $i \in \{0,1\}$), and $\vec{r}_i$ the Bloch vector of $|\psi_i\rangle\langle\psi_i|$. Show that the probability of obtaining outcome $i \in \{0,1\}$ when measuring ρ using μ is given by $\frac{1}{2}(1 + \vec{r} \cdot \vec{r}_i)$. Show that $\vec{r}_0 = -\vec{r}_1$. How can you visualize these two facts on the Bloch sphere?

(d) Now imagine that ρ is an unknown qubit state ρ whose Bloch vector $\vec{r}$ you would like to characterize completely. Consider the following measurement with six outcomes:

$$\mu\colon \{x, y, z\} \times \{0, 1\} \to \mathrm{PSD}(\mathbb{C}^2), \quad \mu(a, b) = \frac{I + (-1)^b \sigma_a}{6},$$

where $\sigma_x = X$, $\sigma_y = Y$, and $\sigma_z = Z$ are the three Pauli matrices. Show that μ is a valid measurement and that the probabilities of measurement outcomes are given by

$$p(a, b) = \frac{1 + (-1)^b r_a}{6}.$$

How can you visualize this formula on the Bloch sphere? Describe how measuring many copies of ρ by using μ allows for estimating the entries of $\vec{r}$ to arbitrary accuracy.