Quantum Information Theory, Spring 2020

Homework problem set #1 **due February 10, 2020**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Monday lecture (in person or by email).

1. *(4 points)* **Trace distance between pure states:** Let $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ be two pure states on an arbitrary Hilbert space $\mathcal{H}$. Show that

$$\frac{1}{2}\|\rho - \sigma\|_1 = \sqrt{1 - |\langle\psi|\phi\rangle|^2}.$$

Here, $\|\cdot\|_1$ denotes the *trace norm*, which for a Hermitian operator A with eigenvalues (a_i) is defined by $\|A\|_1 := \sum_i |a_i|$.

Hint: Argue that the eigenvalues of $\rho - \sigma$ are of the form $(\lambda, -\lambda, 0, \dots, 0)$ for some $\lambda \in \mathbb{R}$. Compute $\|\rho - \sigma\|_1$ and $\mathrm{Tr}[(\rho - \sigma)^2]$ in terms of λ. Can you relate the latter to $|\langle\psi|\phi\rangle|^2$?

2. *(4 points)* **Uncertainty relation:** Given a measurement $\mu\colon \{0,1\} \to \mathrm{PSD}(\mathcal{H})$ with two outcomes and a state $\rho \in \mathrm{D}(\mathcal{H})$, define the *bias* by

$$\beta(\rho) = \big|\mathrm{Tr}[\mu(0)\rho] - \mathrm{Tr}[\mu(1)\rho]\big|.$$

(a) Show that $\beta \in [0,1]$, that $\beta = 1$ iff the measurement outcome is certain, and that $\beta = 0$ iff both outcomes are equally likely (for the given measurement and state).

In class, we discussed how to measure a qubit in the standard basis $|0\rangle$,$|1\rangle$ and in the Hadamard basis $|+\rangle$, $|-\rangle$. Let β_{Std} and β_{Had} denote the bias for these two measurements.

(b) Compute $\beta_{\mathrm{Std}}(\rho)$ and $\beta_{\mathrm{Had}}(\rho)$ in terms of the Bloch vector of the qubit state ρ.

(c) Show that $\beta^2_{\mathrm{Std}}(\rho) + \beta^2_{\mathrm{Had}}(\rho) \leqslant 1$. Why is this called an *uncertainty relation*?

3. *(4 bonus points)* 🖩 **Practice:** In the exercise class, you discussed how to estimate an unknown qubit state ρ by performing the following measurement on many copies of ρ:

$$\mu\colon \{x,y,z\} \times \{0,1\} \to \mathrm{PSD}(\mathbb{C}^2), \quad \mu(a,b) = \frac{I + (-1)^b \sigma_a}{6},$$

where $\sigma_x = X$, $\sigma_y = Y$, and $\sigma_z = Z$ are the Pauli matrices.

The file `01-outcomes.txt` on the course homepage contains $N = 100\,000$ measurement outcomes produced in this way (one per row). Give an estimate for the unknown state ρ.