Formalism of Q. Information Theory

$\boxed{\text{To every (q.) system, we associate a Hilbert space } \mathcal{H}.}$

Complex vector space w/ inner product, complete w.r.t. $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$

This course: $\dim \mathcal{H} < \infty$.

Dirac notation: $\langle\phi|\psi\rangle$ inner product, $|\psi\rangle \in \mathcal{H}$, $\langle\phi| \equiv \langle\phi|\cdot\rangle \in \mathcal{H}^*$ dual space

"bra-ket", "ket", "bra", sends $|\psi\rangle$ to $\langle\phi|\psi\rangle$

$\mathcal{H} = \mathbb{C}^d$: $\quad \sum_i \bar{\phi}_i \psi_i \qquad \begin{pmatrix}\psi_1\\ \vdots \\ \psi_d\end{pmatrix} \qquad (\bar{\phi}_1 \cdots \bar{\phi}_d)$

anti-linear

EX CLASS

* If $\|\psi\|=1$: $|\psi\rangle\langle\psi|$ = orthogonal projection onto $\mathbb{C}|\psi\rangle$

  $\left[|\psi\rangle\langle\psi|\cdot|\psi\rangle = |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{=1} = |\psi\rangle, \text{ while } |\psi\rangle\langle\psi|(|\phi\rangle) = 0 \text{ if } \langle\psi|\phi\rangle = 0\right]$

* Σ finite set $\rightsquigarrow \mathcal{H} = \mathbb{C}^\Sigma$, ONB $\{|x\rangle\}_{x\in\Sigma}$ "standard basis", "computational basis"

  e.g. qubit: $\Sigma = \{0,1\} \rightsquigarrow \mathcal{H} = \mathbb{C}^2$ w/ std basis $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

  bit

Operators on Hilbert Space:

$L(\mathcal{H}) = \{A: \mathcal{H} \to \mathcal{H} \text{ linear}\}$ $\qquad L(\mathcal{H},K) = \{A: \mathcal{H} \to K \text{ linear}\}$

* Adjoint of $A \in L(\mathcal{H},K)$: $A^\dagger \in L(K,\mathcal{H})$ s.th. $\langle\psi|A^\dagger|\phi\rangle = \overline{\langle\phi|A|\psi\rangle}$

  NB: Can think of $\langle\phi| = |\phi\rangle^\dagger$. $\qquad$ in coordinates: $A^\dagger = \bar{A}^T = \overline{A^T}$

* $A \in L(\mathcal{H})$ Hermitian: $A = A^\dagger$.

  real eigenvalues

  $\hookrightarrow$ SPECTRAL THM: $\boxed{A = \sum_i a_i |\psi_i\rangle\langle\psi_i|}$

  orthonormal eigenvectors

* A positive semidefinite (PSD): Hermitian & all eigenvalues ≥ 0

  equ: $A = B^\dagger B$ $\qquad$ equ: $\langle\psi|A|\psi\rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$

* Notation: $A \geq B$ iff $A-B$ is PSD $\qquad PSD(\mathcal{H}) = \{A \in L(\mathcal{H}) \text{ PSD}\}$

$\boxed{\text{A (q.) state on } \mathcal{H} \text{ is an element of } D(\mathcal{H}) = \{ \rho \in PSD(\mathcal{H}),\ \mathrm{tr}[\rho]=1 \}}$

"density operator"

* ρ pure: $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathcal{H}$, $\||\psi\rangle\| = 1$

* maximally mixed state: $\tau = \frac{I}{d}$ $(d = \dim \mathcal{H})$

* ρ classical on $\mathcal{H} = \mathbb{C}^{\Sigma}$: $\rho = \sum_x p_x |x\rangle\langle x|$ prob. dist.

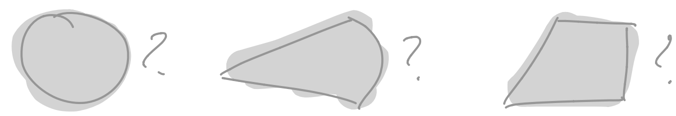
e.g. $\rho = \begin{pmatrix} p_0 & 0 \\ 0 & p_1 \end{pmatrix}$ qubit

* $\underline{\text{SPECTRAL THM:}}$ $\forall \rho\ \exists$ ONB $\{|\psi_i\rangle\}$, prob. dist. $\{p_i\}$: $\boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$

* $D(\mathcal{H})$ is $\underline{\text{convex}}$: $\{p_j, \rho_j\}$ ensemble $\Rightarrow \rho = \sum_j p_j \rho_j$ state

extreme points = pure states.

What does $D(\mathcal{H})$ look like for a qubit?



$\underline{\text{Qubits!}}$ Pauli matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

are basis of 2x2 Herm. matrices (real vector space)

$\Rightarrow \rho = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$ Hermitian, tr $= 1$ $\qquad r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Bloch vector

* eigenvalues: $\{p, 1-p\}$ for $p \in \mathbb{R}$

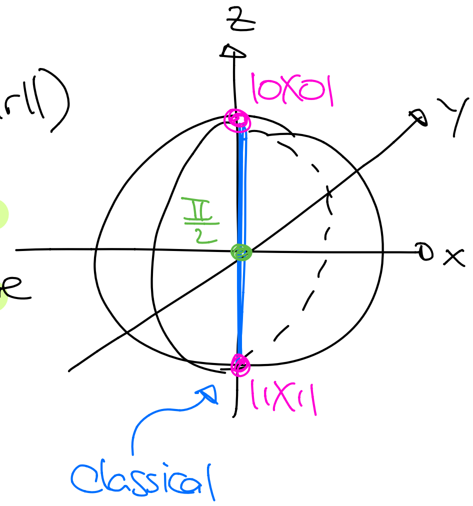
$\Rightarrow p(1-p) = \det(\rho) = \frac{1}{4}(1 - x^2 - y^2 - z^2) = \frac{1}{4}(1 - \|r\|)$

* ρ state: $p(1-p) \geq 0 \iff \|r\| \leq 1$ Bloch ball

* ρ pure: $p(1-p) = 0 \iff \|r\| = 1$ Bloch sphere

* ρ classical: $\rho = \begin{pmatrix} p & \\ & 1-p \end{pmatrix} \cong [0,1] \iff r = \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$

simplex



$\underline{\text{Useful}}$: $x = \mathrm{tr}[X\rho]$, $y = \mathrm{tr}[Y\rho]$, $z = \mathrm{tr}[Z\rho]$.

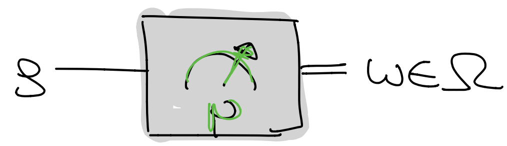
$\rightarrow$ $\boxed{\text{EX CLASS}}$

How do we get information about a q. state?

A measurement (POVM) on $\mathcal{H}$ with outcomes in Ω (finite set) is

$$\mu : \Omega \to \mathrm{PSD}(\mathcal{H}) \quad \text{s.th.} \quad \sum_{\omega \in \Omega} \underbrace{\mu(\omega)}_{\text{"POVM element"}} = I_{\mathcal{H}}$$

Rule: If measure in state ρ, obtain outcome $\omega \in \Omega$ with probability $p(\omega) = \mathrm{tr}[\mu(\omega)\rho]$ prob. dist!



NB: This does NOT specify what happens to the q. state! → LATER

* μ projective: $\mu(\omega)$ orthogonal projection $\forall \omega$
* basis measurement: $\mu(\omega) = |\psi_\omega\rangle\langle\psi_\omega|$ for ONB $\{|\psi_\omega\rangle\}_{\omega\in\Omega}$ of $\mathcal{H}$

  e.g. Standard basis measurement for $\mathcal{H} = \mathbb{C}^\Sigma$:

  $$\mu : \Sigma \to \mathrm{Pos}(\mathcal{H}), \quad x \mapsto |x\rangle\langle x|$$

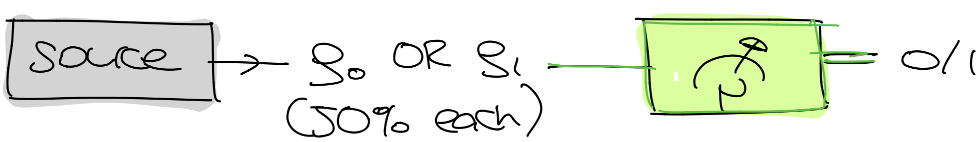
Qubits: Can measure in standard basis $\{|0\rangle, |1\rangle\}$, but also in Hadamard basis $\{|+\rangle, |-\rangle\}$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

e.g. $\rho = \begin{pmatrix} 2/3 & \\ & 1/3 \end{pmatrix}$:

$$\left.\begin{aligned} P_{\mathrm{Std}}(0) &= \mathrm{tr}[|0\rangle\langle 0|\rho] = \langle 0|\rho|0\rangle = \tfrac{2}{3} \\ P_{\mathrm{Had}}(+) &= \mathrm{tr}[|+\rangle\langle +|\rho] = \langle +|\rho|+\rangle = \tfrac{1}{2} \end{aligned}\right\} \text{CANNOT BOTH BE} = 1\ !$$

→ HOMEWORK

Discriminating q. states:



WANT: Measurement $\mu : \{0,1\} \to \mathrm{Pos}(\mathcal{H})$ that maximizes

$$P_{\mathrm{success}} = \tfrac{1}{2}\mathrm{tr}[\mu(0)\rho_0] + \tfrac{1}{2}\mathrm{tr}[\mu(1)\rho_1]$$

How to find it? → Next week!