Quantum Information Theory, Spring 2019

Problem Set 8 **due April 1, 2019**

1. *(4 points)* **Measurements and trace distance:** In this problem, you will revisit how to distinguish quantum states by using measurements. Given states $\rho, \sigma \in D(\mathcal{X})$ and a measurement $\mu\colon \Gamma \to \mathrm{Pos}(\mathcal{X})$, let $p, q \in \mathcal{P}(\Gamma)$ denote the corresponding distributions of measurement outcomes.

   (a) Prove that $\|p - q\|_1 \leq \|\rho - \sigma\|_1$.

   (b) Show that, for every ρ and σ, there exists a measurement μ such that equality holds.

   Hint: In Lecture 1, we discussed how to optimally distinguish ρ and σ.

2. *(4 points)* **Holevo χ-quantity:** Alice wants to communicate a classical message to Bob by sending a quantum state. She chooses one state $\rho_x \in D(\mathcal{Y})$ for each possible message $x \in \Sigma$ that she may want to send, and Bob chooses a measurement $\mu\colon \Sigma \to \mathrm{Pos}(\mathcal{Y})$ that he uses to decode.

   (a) Write down a formula for the probability that Bob successfully decodes the message if the message is drawn according to an arbitrary probability distribution $p \in \mathcal{P}(\Sigma)$.

   In class, we used the Holevo bound to prove that if this probability is 100% then, necessarily, the Holevo χ-quantity of the ensemble $\{p_x, \rho_x\}$ must be equal to $H(p)$.

   (b) Show that this condition is also sufficient: If $\chi(\{p_x, \rho_x\}) = H(p)$ then there exists a measurement μ such that Bob decodes the message with 100% probability of success.

   Hint: In class we discussed when an ensemble satisfies $\chi(\{p_x, \rho_x\}) = H(p)$.

3. *(4 points)* **Properties of relative entropy:** Prove the following two properties of the quantum relative entropy by using its monotonicity property:

   (a) *Entropy increase:* $H(\Phi[\rho]) \geq H(\rho)$ for every $\rho \in D(\mathcal{X})$ and *unital* channel $\Phi \in C(\mathcal{X}, \mathcal{Y})$. Recall that a channel is *unital* if $\Phi[I_X] = I_Y$.

   (b) *Joint convexity:* $D(\sum_{x \in \Sigma} p_x \rho_x \| \sum_{x \in \Sigma} p_x \sigma_x) \leq \sum_{x \in \Sigma} p_x D(\rho_x \| \sigma_x)$, where $(p_x)_{x \in \Sigma}$ is an arbitrary finite probability distribution and $(\rho_x)_{x \in \Sigma}$, $(\sigma_x)_{x \in \Sigma}$ families of states in $D(\mathcal{Y})$. You may assume that the operators ρ_x and σ_x are positive definite.

   Hint: In the exercise class, we computed the logarithm of a cq-state.

4. *(4 points)* **Practice:** In this problem, you will compute some relative entropies and verify the monotonicity property. Let $\rho = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}$ and $\sigma = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

   (a) Compute the relative entropies $D(\rho \| \sigma)$ and $D(\sigma \| \rho)$.

   (b) Compute the relative entropies $D(\mathcal{M}[\rho] \| \mathcal{M}[\sigma])$ and $D(\mathcal{M}[\sigma] \| \mathcal{M}[\rho])$, where the channel $\mathcal{M}[\omega] = \sum_x \langle x|\omega|x\rangle |x\rangle\langle x|$ correponds to a standard basis measurement.