Quantum Information Theory, Spring 2019

Problem Set 7 **due March 25, 2019**

1. *(2 points)* **Maximal mutual information I:** In class, we proved that $I(X:Y) \leq 2\log d$ for every state $\rho_{XY} \in D(\mathcal{X} \otimes \mathcal{Y})$ with $\mathcal{X} = \mathcal{Y} = \mathbb{C}^d$. Show that $I(X:Y) = 2\log d$ if and only if ρ_{XY} is a pure state with $\rho_X = \rho_Y = I/d$ (such quantum states are called *maximally entangled*). Write down the Schmidt decomposition of a general state of this form.

2. *(2 points)* **Maximal mutual information II:** In the exercises, we proved that $I(X:Y) \leq \log d$ for every distribution $p_{XY} \in \mathcal{P}(\Sigma \times \Gamma)$ with $|\Sigma| = |\Gamma| = d$. Show that $I(X:Y) = \log d$ if and only if p_{XY} is *maximally correlated*, i.e., of the form $p_{XY}(x,y) = \frac{1}{d}\delta_{f(x),y}$ for a bijection $f\colon \Sigma \to \Gamma$.

   Hint: In the exercise class we characterized the distributions with $H(XY) = H(X)$.

3. *(8 points)* **Entropic uncertainty relation:** In this problem, you will prove another uncertainty relation. Let $\rho \in D(\mathbb{C}^2)$ and denote by p_{std} and p_{Had} the probability distributions of outcomes when measuring ρ in the standard basis and Hadamard basis, respectively. You will show that:

$$H(p_{\text{std}}) + H(p_{\text{Had}}) \geq H(\rho) + 1 \tag{1}$$

   (a) Why is it appropriate to call (1) an *uncertainty relation*?
   (b) Find a state ρ for which the uncertainty relation is saturated (i.e., an equality).

   To start, recall the Pauli matrices $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   (c) Show that $\frac{1}{2}(\rho + Z\rho Z) = \begin{pmatrix} \langle 0|\rho|0\rangle & 0 \\ 0 & \langle 1|\rho|1\rangle \end{pmatrix}$ and deduce that $H(p_{\text{std}}) = H(\frac{1}{2}(\rho + Z\rho Z))$.
   (d) Show that, similarly, $H(p_{\text{Had}}) = H(\frac{1}{2}(\rho + X\rho X))$.

   Now consider the following three-qubit state,

$$\omega_{ABC} = \frac{1}{4}\sum_{a=0}^{1}\sum_{c=0}^{1} |a\rangle\langle a| \otimes X^a Z^c \rho Z^c X^a \otimes |c\rangle\langle c|,$$

   where we denote the three subsystems by A, B, C (to avoid confusion with the Pauli matrices) and abbreviate $X^0 = I$, $X^1 = X$ and $Z^0 = I$, $Z^1 = Z$. Note that subsystems A, C are classical.

   (e) Show that $H(ABC) = 2 + H(\rho)$. Use parts (c) and (d) to verify that $H(AB) = 1 + H(p_{\text{std}})$, $H(BC) = 1 + H(p_{\text{Had}})$, and $H(B) = 1$ in state ω_{ABC}.
   (f) Use part (e) and the strong subadditivity inequality to deduce (1).

4. *(4 points)* **Practice:** In this problem you can verify an instance of the Holevo bound. Consider the ensemble $\{p_x, \rho_x\}_{x\in\{0,1\}}$ where $p_0 = p_1 = \frac{1}{2}$ and $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = 0.2|+\rangle\langle +| + 0.8|-\rangle\langle -|$.

   (a) Compute the Holevo χ-quantity for this ensemble.
   (b) Let $\mu\colon \{0,1\} \to \text{Pos}(\mathcal{X})$ denote the *pretty good measurement* for $\{\rho_x\}$ defined in Problem 4.1. Compute $I(X:Z)$ for $p_{XZ}(x,z) = p_x \operatorname{Tr}[\rho_x \mu(z)]$. Verify that $I(X:Z) \leq \chi$.