# Quantum Information Theory, Spring 2019

**Problem Set 5** **due March 11, 2019**

1. *(4 points)* **Optimality of the Shannon entropy:** In this problem, you will show that it is impossible to compress at rates below the Shannon entropy of the source. Fix $\delta \in (0,1)$. Recall from class that an $(n, R, \delta)$-*code* for a distribution $p \in \mathcal{P}(\Sigma)$ consists of functions $\mathcal{E}_n \colon \Sigma^n \to \{0,1\}^{\lfloor nR \rfloor}$ and $\mathcal{D}_n \colon \{0,1\}^{\lfloor nR \rfloor} \to \Sigma^n$ such that $\sum_{x : \mathcal{D}_n(\mathcal{E}_n(x)) = x} p(x_1) \cdots p(x_n) \geq 1 - \delta$.

   (a) Show that any $(n, R, \delta)$-code transmits at most $2^{nR}$ sequences of length $n$ correctly.
   (b) Show that, if $R < H(p)$, $(n, R, \delta)$-codes can only exist for finitely many values of $n$.
   *Hint: Distinguish between typical and atypical sequences.*

2. *(4 points)* **Universal classical data compression:** In this problem, you will show that there exist compression protocols that work for any data source with entropy less than some $R > 0$. For simplicity, you may assume that the source emits a single bit at a time ($\Sigma = \{0,1\}$).

   Thus, construct functions $\mathcal{E}_n \colon \Sigma^n \to \{0,1\}^{\lfloor nR \rfloor}$ and $\mathcal{D}_n \colon \{0,1\}^{\lfloor nR \rfloor} \to \Sigma^n$ for $n = 1, 2, \dots$ that satisfy the following property: For every probability distribution $p \in \mathcal{P}(\Sigma)$ with $H(p) < R$, it holds that $\sum_{x : \mathcal{D}_n(\mathcal{E}_n(x)) = x} p(x_1) \cdots p(x_n) \to 1$ as $n \to \infty$.

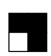
   *Hint: Adapt the procedure used in class for compressing coin flips.*

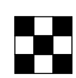
3. *(4 points)* **Gentle measurement lemma:** In this problem, you will derive a useful technical result known as the *gentle measurement lemma*. It asserts that if $\rho \in D(\mathcal{X})$ is a state and $0 \leq Q \leq I$ an operator such that $\mathrm{Tr}[Q\rho] \geq 1 - \varepsilon$, then the following two inequalities hold:

$$F\Big(\rho, \frac{\sqrt{Q}\rho\sqrt{Q}}{\mathrm{Tr}[Q\rho]}\Big) \geq \sqrt{1-\varepsilon} \quad \text{and} \quad \frac{1}{2}\Big\|\rho - \frac{\sqrt{Q}\rho\sqrt{Q}}{\mathrm{Tr}[Q\rho]}\Big\|_1 \leq \sqrt{\varepsilon}$$

   (a) Prove that $\mathrm{Tr}\sqrt{\sqrt{\rho}\sqrt{Q}\rho\sqrt{Q}\sqrt{\rho}} = \mathrm{Tr}[\sqrt{Q}\rho]$ and deduce the first inequality from this.
   (b) Deduce the second inequality from the first by using an inequality from the exercise class.
   (c) Explain in one or two sentences why this result is known as the *gentle measurement lemma*.

4. *(4 points)* **Practice:** A binary image of size $n \times n$ can be represented by a bitstring, where we list the pixel values (`0`=black pixel, `1`=white pixel) row by row, starting with the first row. We can then compress the bitstring in the following way: First, compute the number $k$ of ones in the bitstring. Next, compute the index $m \in \{0, 1, \dots, \binom{n^2}{k} - 1\}$ of the bitstring in the lexicographically sorted list of all bitstrings of length $n^2$ that contain $k$ ones. The triple $(n, k, m)$ defines the compression of the image.

   For example, the $2 \times 2$-image  corresponds to the bitstring `0010`. There are four strings with $k = 1$ ones. In lexicographic order, they are `0001`, `0010`, `0100`, and `1000`. The index of our bitstring in this list is $m = 1$. Thus, we would compress this picture by the triple $(2, 1, 1)$.

   (a) What is the bitstring corresponding to the following image? What is its compression? 
   (b) Can you decompress the image given by $(n, k, m) = (8, 8, 986860893)$?