Quantum Information Theory, Spring 2019

Problem Set 4 **due March 4, 2019**

1. *(3 points)* **Pretty good measurement:** Let $\mathcal{R} = \{\rho_a : a \in \Sigma\} \subseteq \mathrm{D}(\mathcal{X})$ be a collection of density matrices such that $\mathrm{span}(\mathcal{R}) = \mathrm{L}(\mathcal{X})$. Let

$$Q = \sum_{a\in\Sigma} \rho_a. \tag{1}$$

   (a) Show that Q is positive *definite* (i.e., $Q > 0$, meaning that all eigenvalues of Q are strictly positive) and thus Q is invertible.
   Hint: Assume that Q has an eigenvector $|\psi\rangle \neq 0$ with eigenvalue 0. Use $|\psi\rangle\langle\psi|$ and the span property to get a contradiction.

   (b) Define $\mu : \Sigma \to \mathrm{L}(\mathcal{X})$ via

$$\mu(a) = Q^{-1/2} \rho_a Q^{-1/2}. \tag{2}$$

   Show that μ is a measurement.

2. *(2 points)* **Measuring the maximally entangled state:** Let

$$|\Psi\rangle_{\mathsf{XY}} = \frac{1}{\sqrt{|\Sigma|}} \sum_{a\in\Sigma} |a\rangle_{\mathsf{X}} \otimes |a\rangle_{\mathsf{Y}}$$

   be the maximally entangled state on registers XY with underlying spaces $\mathcal{X} = \mathcal{Y} = \mathbb{C}^\Sigma$. Let $\mu : \Gamma \to \mathrm{Pos}(\mathcal{X})$ be an arbitrary measurement with outcome set Γ.

   (a) Compute the probability of outcome $g \in \Gamma$ when measuring the X register of $|\Psi\rangle_{\mathsf{XY}}$ with μ.

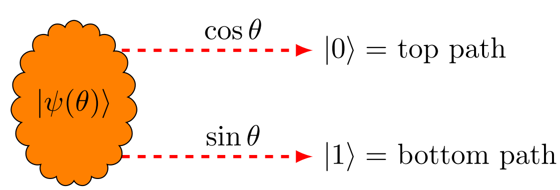
   (b) Compute the state on the remaining register Y after the measurement μ is performed on the first register and has produced outcome $g \in \Gamma$.

3. *(7 points)* **Elitzur–Vaidman bomb tester:** You are given a suitcase that is either empty or contains a very sensitive bomb that would explode even if hit by a single photon of light. Your job is to determine whether the suitcase is empty or not (of course, without exploding the bomb…). This can be done using a clever quantum-mechanical trick known as *interaction-free measurement*. To understand how it works, let us first rephrase the problem mathematically.

   - A single photon of light can be described by a quantum state

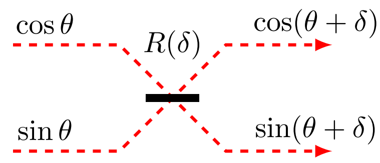
$$|\psi(\theta)\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos\theta|0\rangle + \sin\theta|1\rangle, \tag{3}$$

     for some $\theta \in \mathbb{R}$, where $|0\rangle$ and $|1\rangle$ represent two possible paths the photon can take:



For example, the state $|\psi(0)\rangle = |0\rangle$ represents a photon that is fully on the top path while $|\psi(\pi/2)\rangle = |1\rangle$ corresponds to a photon fully on the bottom path. In general, a photon can take both paths simultaneously in superposition. This corresponds to a general state $|\psi(\theta)\rangle$ in equation (3).

- You can adjust the amplitudes on the two paths by using a beam splitter $R(\delta)$:



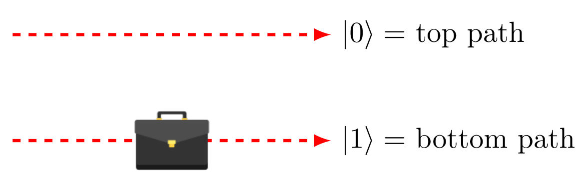
  This has the effect of modifying the photon's state as follows:

$$R(\delta)|\psi(\theta)\rangle = |\psi(\theta+\delta)\rangle,$$

  where $\delta \in \mathbb{R}$ is a parameter describing the beam splitter. Note that $R(\delta)$ corresponds to the following orthogonal matrix:

$$R(\delta) = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}.$$

- To test whether the suitcase contains a bomb, you can make two tiny holes at opposite sides of the suitcase and place it on one of the two paths, say the bottom one:

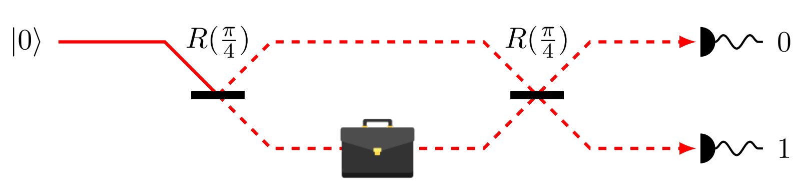


  If the bomb is present, this has the effect of measuring the state of the photon in the $\{|0\rangle, |1\rangle\}$ basis by a *non-destructive* projective measurement:

  – the outcome "1" means the photon was on the bottom path (this triggers the bomb and kills you),
  – the outcome "0" means the photon was on the top path and hence did not interact with the bomb (in this case you are lucky and the bomb does not explode).

  If there was no bomb inside the suitcase, no measurement happens and the state of the photon is not disturbed in any way.

The full bomb testing procedure looks as follows:



The photon starts out on the top path, i.e., in the state $|0\rangle = |\psi(0)\rangle$. It then passes through a $\delta = \pi/4$ beam splitter which sends it on to two possible paths in superposition. The bottom path goes through the suitcase, potentially resulting in a measurement due to the bomb. The two paths are then recombined by another $\delta = \pi/4$ beam splitter, and the photon is measured at the end: the two black semicircles denote photodetectors that measure the photon in the $\{|0\rangle, |1\rangle\}$ basis by a *destructive* projective measurement.

(a) Assuming there was no bomb, derive the state $|\psi(\theta)\rangle$ before the final measurement.
(b) Assuming there was a bomb, what is the probability of triggering the bomb?
(c) Assuming there was a bomb and it did not explode, derive the state $|\psi(\theta')\rangle$ before the final measurement.
(d) Assume you run the experiment without knowing if the bomb is present. If no explosion occurred, what can you conclude from the outcome $a \in \{0, 1\}$ of the final measurement? I.e., what do you learn if the outcome was 1 and what do you learn if the outcome was 0?
(e) Consider the following more complicated experiment. Let the initial state be $|0\rangle$ as before. Pick some integer $n \geq 1$ and repeat the following two steps n times:
- Apply the beam splitter $R(\delta)$ with $\delta = \pi/(2n)$.
- Let the bottom path go through the suitcase.

The final state is then measured in the standard basis using photodetectors as before. Analyze this experiment using the following steps:

i. What is the final state if there was no bomb?
ii. What is the final state if there was a bomb but you were very lucky and it did not explode in any of the n trials?
iii. Assuming you are still alive, how can you tell from the final measurement whether there was a bomb or not?
iv. Assuming the bomb is present, compute the probability of you still begin alive as a function of n. What value of n should you choose so that you are still alive with probability 90%?

Hint: Don't look!

4. *(4 points)* **Practice:** The files `pset4-R1.txt`, `pset4-R2.txt`, `pset4-R3.txt` contain the entries of three 5×5 density matrices $\rho_1, \rho_2, \rho_3 \in \mathrm{L}(\mathcal{X})$ where $\mathcal{X} = \mathbb{C}^5$. These matrices define a measurement with outcomes $\Sigma = \{1, 2, 3\}$ according to equations (1) and (2). For each $i \in \Sigma$, let $p_i = (p_i(1), p_i(2), p_i(3))$ be the probability distribution obtained by measuring the state ρ_i with this measurement, i.e.,

$$p_i(a) = \mathrm{Tr}\big[\mu(a)\rho_i\big],$$

for each outcome $a \in \Sigma$. Compute the three probability distributions p_1, p_2, p_3.

Hint: On the course homepage you can find instructions for loading a complex matrix.