Quantum Information Theory, Spring 2019

Problem Set 13 **due May 13, 2019**

1. *(4 points)* **Rényi-2 entropy:** In this problem you will study a new entropy measure called the *Rényi-2 entropy*. It is defined by $H_2(\rho) := -\log \operatorname{Tr}[\rho^2]$ for any quantum state $\rho \in \mathrm{D}(\mathcal{X})$.

   (a) Find a formula for $H_2(\rho)$ in terms of the eigenvalues of ρ.
   (b) Show that $H_2(\rho) \leq H(\rho)$ by using Jensen's inequality.
   (c) Show that $\operatorname{Tr}[\rho^2] = \operatorname{Tr}[F\rho^{\otimes 2}]$, where F is the *swap operator* sending $|x_1, x_2\rangle \mapsto |x_2, x_1\rangle$.

2. *(4 points)* **Average entanglement:** In this exercise you will study the average entanglement of a random pure state in $\mathcal{X} \otimes \mathcal{Y}$ drawn from the probability distribution $d\psi_{XY}$ discussed in class. Recall that the entanglement entropy of a pure state ψ_{XY} is given by $H(\psi_X) = H(\psi_Y)$.

   (a) Let F_{XX}, F_{YY} denote the swap operators on $\mathcal{X}^{\otimes 2}$, $\mathcal{Y}^{\otimes 2}$ and let $d_X = \dim \mathcal{X}$, $d_Y = \dim \mathcal{Y}$. Use the integral formula for the symmetric subspace to deduce that

   $$\int |\psi_{XY}\rangle^{\otimes 2} \langle \psi_{XY}|^{\otimes 2} \, d\psi_{XY} = \frac{1}{d_X d_Y (d_X d_Y + 1)} \left(I_{XX} \otimes I_{YY} + F_{XX} \otimes F_{YY} \right).$$

   (b) Verify that $\int \operatorname{Tr}[\psi_X^2] \, d\psi_{XY} = \frac{d_X + d_Y}{d_X d_Y + 1}$.
   (c) Show that the average Rényi-2 entropy $H_2(\psi_X)$ for a random pure state ψ_{XY} is at least $\log(\min(d_X, d_Y)) - 1$. Conclude that the same holds for the entanglement entropy.

   Hint: Use Problem 1 and Jensen's inequality.

3. *(3 points)* **Haar measure:** In the exercise class, we discussed the Haar measure on $\mathrm{U}(\mathcal{X})$, which is the unique probability measure dU with the following property: For every continuous function f on $\mathrm{U}(\mathcal{X})$ and for all unitaries $V, W \in \mathrm{U}(\mathcal{X})$, it holds that $\int f(U) \, dU = \int f(VUW) \, dU$.

   (a) Argue that, for any operator $A \in L(\mathcal{X}^{\otimes n})$, the so-called *twirl* $\int U^{\otimes n} A U^{\dagger, \otimes n} \, dU$ can always be written as a linear combination of permutation operators R_π, $\pi \in S_n$.
   (b) Deduce that $\int U^{\otimes 2} A (U^\dagger)^{\otimes 2} \, dU = \alpha I + \beta F$ for every $A \in L(\mathcal{X}^{\otimes 2})$, where F is the swap operator on $\mathcal{X}^{\otimes 2}$, $\alpha = \frac{d}{d^3 - d} \operatorname{Tr}[A] - \frac{1}{d^3 - d} \operatorname{Tr}[FA]$, and $\beta = \frac{d}{d^3 - d} \operatorname{Tr}[FA] - \frac{1}{d^3 - d} \operatorname{Tr}[A]$.

4. *(5 points)* **Practice:** In the exercises, we discussed an application of de Finetti theorem in quantum physics. Let us numerically verify our results for the following operator on $(\mathbb{C}^2)^{\otimes n}$:

   $$H = -\frac{1}{n-1} \sum_{i \neq j} Z_i Z_j - \sum_{i=1}^{n} X_i,$$

   where $Z_i = I^{\otimes (i-1)} \otimes Z \otimes I^{\otimes (n-i)}$ and likewise for X_i. As usual, $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

   (a) Compute the smallest eigenvalue $E_0(n)$ of H for $n = 2, \ldots, 10$ (or as high as you can go), and plot the corresponding energy densities $E_0(n)/n$.
   (b) Find an operator h on $\mathbb{C}^2 \otimes \mathbb{C}^2$ such that $H = \frac{1}{n-1} \sum_{i \neq j} h_{i,j}$ (notation as in exercise class), minimize $\langle \psi^{\otimes 2} | h | \psi^{\otimes 2} \rangle$ over all unit vectors $|\psi\rangle$ (in any way you like), and compare with (a).

   Python hint: You can use `numpy.kron` to compute the tensor product of operators.