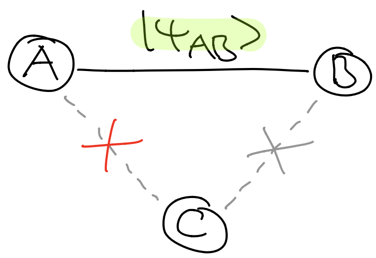
Monogamy of Entanglement

Watrous §7.2

LAST WEEKS

Next two lectures: Study entanglement + communication via probabilistic method



ρ_{AB} pure $\overset{\text{PSET 2}}{\Longrightarrow}$ $\rho_{ABC} = \rho_{AB} \otimes \rho_C$

$\Rightarrow \rho_{AC} = \rho_A \otimes \rho_C$ and $\rho_{BC} = \rho_B \otimes \rho_C$

$\boxed{\text{Cannot share } \underline{\text{pure}} \text{ entangled state w/ more than one system!}}$ MONOGAMY

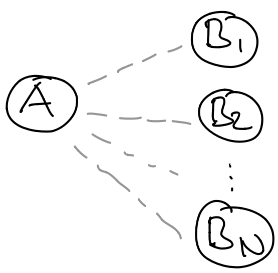
* in contrast: classical correlations can be shared:

$$\rho_{ABC} = \tfrac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right) \rightsquigarrow \rho_{AB} = \rho_{AC} = \tfrac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$|\psi_{ABC}\rangle = \tfrac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \quad \overset{!!!}{\cancel{\rightsquigarrow}} \quad |\phi^+\rangle = \tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

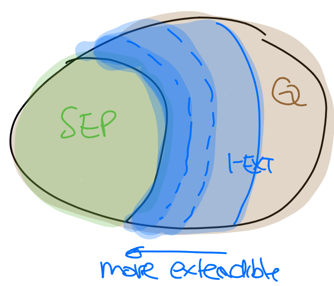
Beyond pure states? quantitative?

* If a state ρ_{AB_1} can be extended to $\rho_{AB_1 \ldots B_N}$ s.th. $\rho_{AB_1} = \ldots = \rho_{AB_N}$ then ρ_{AB_1} "not very" entangled.

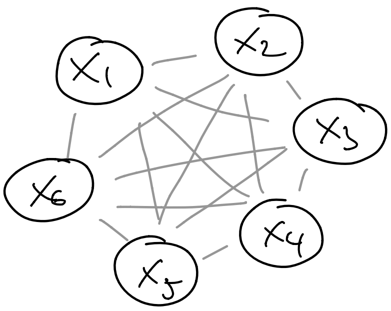


$$\rho_{AB_1} \approx \boxed{\sum_i p_i \, \rho_A^i \otimes \rho_B^i}$$

separable



* If $\rho_{X_1 \ldots X_N}$ is such that all $\rho_{X_i X_j}$ same then $\rho_{X_i X_j}$ "not very" entangled.



De Finetti Theorem: If $N \gg 1$,

$$\rho_{X_1 X_2} \approx \boxed{\int d\sigma \, p(\sigma) \, \sigma^{\otimes 2}}$$

mixture of tensor powers $\Rightarrow$ separable!

Ex: n bosons

Ex: Mean-field theory: $H = \sum_{i<j} h_{ij} \rightsquigarrow |E_0\rangle$

same interaction

if non-degenerate

We will prove de Finetti for the ...

Symmetric subspace:

$$\mathrm{Sym}^n(\mathcal{H}) = \{ |\Phi\rangle \in \mathcal{H}^{\otimes n} : R_\pi |\Phi\rangle = |\Phi\rangle \ \forall \pi \in S_n \}$$

where we used the permutation op's

$$R_\pi : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}, \quad R_\pi |\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle = |\psi_{\pi^{-1}(1)}\rangle \otimes \dots \otimes |\psi_{\pi^{-1}(n)}\rangle$$

* $|\Phi\rangle = |\psi\rangle^{\otimes n} \in \mathrm{Sym}^n(\mathcal{H}) \quad \forall |\psi\rangle \in \mathcal{H}$

* e.g. $\mathrm{Sym}^2(\mathbb{C}^2)$ spanned by $|00\rangle, |11\rangle, \frac{|01\rangle + |10\rangle}{\sqrt{2}}$

  Missing: $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$ anti-symmetric!

* For $|\Phi\rangle \in \mathrm{Sym}^n(\mathcal{H})$: All $\Phi_{x_i x_j}$ same!

EX: proof of these facts

Symmetrizer: $\Pi_n = \frac{1}{n!} \sum_{\pi \in S_n} R_\pi$ orthogonal projection onto $\mathrm{Sym}^n(\mathcal{H}) \subseteq \mathcal{H}^{\otimes n}$

- $|\Phi\rangle$ arbitrary $\Rightarrow \Pi_n |\Phi\rangle \in \mathrm{Sym}^n(\mathcal{H})$
- $|\Phi\rangle \in \mathrm{Sym}^n(\mathcal{H}) \Rightarrow \Pi_n |\Phi\rangle = |\Phi\rangle$
- $\Pi_n^\dagger = \Pi_n$

Pf: Use $R_\pi R_\tau = R_{\pi\tau}$ and $R_\pi^\dagger = R_{\pi^{-1}}$. ☺

e.g. $\Pi_2 = \frac{1}{2}(I + F)$ where $F = R_{1 \leftrightarrow 2}$ swap operator

Basis of $\mathrm{Sym}^n(\mathbb{C}^d)$: $\Pi_n |\underbrace{1 \dots 1}_{t_1} \underbrace{2 \dots 2}_{t_2} \dots \underbrace{d \dots d}_{t_d}\rangle$ for $t = (t_1, \dots, t_d)$, $\sum_i t_i = n$

type, occup. numbers

= superposition of basis vectors with t_1 ones, t_2 twos, ..

$$\Rightarrow \dim \mathrm{Sym}^n(\mathbb{C}^d) = \binom{n+d-1}{n} = \frac{(n+d-1)!}{n!\,(d-1)!}$$

Integral formula: $$\Pi_n = \binom{n+d-1}{n} \int d\psi \, |\psi\rangle\langle\psi|^{\otimes n}$$ Proof? Later...

"uniform" probability measure on pure states

pure states

for qubits: {pure states} = S^2 ↳ usual measure on S^2 ☺

unique probability measure on pure states on $\mathcal{H}$ invariant under $|\psi\rangle \mapsto U|\psi\rangle$

i.e. $\int d\psi \, f(|\psi\rangle\langle\psi|) = \int d\psi \, f(U|\psi\rangle\langle\psi|U^\dagger) \quad \forall U \in \mathcal{U}(\mathcal{H})$

$$\boxed{\begin{array}{l}\underline{\text{De Finetti Theorem}}: \ |\Phi\rangle \in \mathrm{Sym}^{k+n}(\mathbb{C}^d) \Rightarrow \exists \text{ density } p(\psi): \\ \frac{1}{2}\left\| \Phi_{X_1\cdots X_k} - \int d\psi\, p(\psi)\, |\psi\rangle^{\otimes k}\langle\psi|^{\otimes k} \right\|_1 \le \sqrt{\frac{dk}{n+k}}\end{array}}$$

Can be generalized to arbitrary states $\rho_{X_1\cdots X_n}$ s.th. $\rho_{X_{i_1}\cdots X_{i_k}}$ same.

Proof:

$$\Phi_{X_1\cdots X_k} = \mathrm{tr}_{X_{k+1}\cdots X_{k+n}}[\Phi] \overset{\text{symmetry}}{=} \mathrm{tr}_{X_{k+1}\cdots X_{k+n}}\left[\overbrace{(I_k \otimes \Pi_n)\,|\Phi\rangle}^{=|\Phi\rangle\,!}\langle\Phi|\right]$$

$$= \binom{n+d-1}{n}\int d\psi\ \mathrm{tr}_{X_{k+1}\cdots X_{k+n}}\left[(I_k \otimes |\psi\rangle^{\otimes n}\langle\psi|^{\otimes n})\,|\Phi\rangle\langle\Phi|\right]$$

$$\overset{?}{=} \binom{n+d-1}{n}\int d\psi\ (I_k \otimes \langle\psi|^{\otimes n})\,|\Phi\rangle\langle\Phi|\,(I_k \otimes |\psi\rangle^{\otimes n})$$

$$= \int d\psi\ p(\psi)\ |V_\psi\rangle\langle V_\psi|$$

where we defined $\sqrt{p(\psi)}\,|V_\psi\rangle = \binom{n+d-1}{n}(I_k \otimes \langle\psi|^{\otimes n})\,|\Phi\rangle$

prob. density unit vector

Let's compare this with $\tilde{\Phi}_{X_1\cdots X_k} := \int d\psi\ p(\psi)\ |\psi\rangle^{\otimes k}\langle\psi|^{\otimes k}$:

$$\frac{1}{2}\left\|\Phi_{X_1\cdots X_k} - \tilde{\Phi}_{X_1\cdots X_k}\right\|_1 \le \int d\psi\ p(\psi)\ \frac{1}{2}\left\|\, |V_\psi\rangle\langle V_\psi| - |\psi\rangle^{\otimes k}\langle\psi|^{\otimes k}\right\|_1$$

$$= \int d\psi\ p(\psi)\sqrt{1-|\langle V_\psi|\psi^{\otimes k}\rangle|^2} \overset{\text{Jensen}}{\le} \sqrt{\int d\psi\ p(\psi)\left(1-|\langle V_\psi|\psi^{\otimes k}\rangle|^2\right)}$$

$$= \sqrt{1-\underbrace{\int d\psi\ p(\psi)\ |\langle V_\psi|\psi^{\otimes k}\rangle|^2}} \le \sqrt{\frac{dk}{N}}.$$

$$= \int d\psi\ p(\psi)\ \langle V_\psi|\psi^{\otimes k}\rangle\langle\psi^{\otimes k}|V_\psi\rangle = \binom{n+d-1}{n}\int d\psi\ \langle\Phi|\psi^{\otimes(n+k)}\rangle\langle\ldots\rangle$$

$$= \binom{n+d-1}{n}\binom{n+k+d-1}{n+k}^{-1}\underbrace{\langle\Phi|\Pi_{n+k}|\Phi\rangle}_{=1\ \text{symmetry!}} = \frac{(n+d-1)!}{n!\,\cancel{(d-1)!}}\ \frac{(n+k)!\,\cancel{(d-1)!}}{(n+k+d-1)!}$$

$$= \frac{(n+d-1)\cdots(n+1)}{(n+k+d-1)\cdots(n+k+1)} \ge \left(\frac{n+1}{n+k+1}\right)^{d-1} = 1-\frac{(d-1)k}{n+k+1} \ge 1-\frac{dk}{n+k}$$

$\square$

We discussed the following in the $\boxed{\text{exercise class:}}$

On the integral formula: The integral formula follows from the following important fact:

$$\boxed{\begin{array}{l}\text{FACT: For } A \in L(\mathcal{H}^{\otimes n}): \\ U^{\otimes n} A U^{\dagger \otimes n} = A \quad \forall U \in \mathcal{U}(\mathcal{H}) \Rightarrow A \in \operatorname{span}\{R_\pi : \pi \in S_n\}\end{array}}$$

!

Proof: We want to show that

$$\Pi_n = \frac{1}{n!}\sum_\pi R_\pi \overset{?}{=} \tilde{\Pi}_n := \binom{n+d-1}{n} \int d\psi \, |\psi\rangle^{\otimes n} \langle\psi|^{\otimes n}$$

Note:

$$\tilde{\Pi}_n = U^{\otimes n} \tilde{\Pi}_n U^{\dagger, \otimes n} \quad (\forall U)$$

inv. under $|\psi\rangle \mapsto U|\psi\rangle$

FACT

$$\Rightarrow \tilde{\Pi}_n = \sum_\pi c_\pi R_\pi \text{ for certain } c_\pi \in \mathbb{C}$$

$$\Rightarrow \tilde{\Pi}_n = \Pi_n \tilde{\Pi}_n = \frac{1}{n!}\sum_\pi R_\pi \sum_\tau c_\tau R_\tau = \frac{1}{n!}\sum_{\pi,\tau} c_\tau R_{\pi\tau}$$

$\Pi_n |\psi\rangle^{\otimes n} = |\psi\rangle^{\otimes n}$

arbitrary permutation γ for fixed τ

$$= \frac{1}{n!}\sum_{\tau,\gamma} c_\tau R_\gamma = \Big(\sum_\tau c_\tau\Big)\Pi_n, \text{ i.e. } \boxed{\tilde{\Pi}_n \propto \Pi_n}$$

$\rightsquigarrow$ "=" since $\operatorname{tr}[\tilde{\Pi}_n] = \binom{n+d-1}{n} = \operatorname{tr}[\Pi_n]$.

$\square$