Formalism of Q. Information Theory

§2, §3.1 in Watrous

$\boxed{\text{To every (q.) system } X \text{, we associate a Hilbert space } \mathcal{H}.}$

* This course: $\dim \mathcal{H} < \infty$

adjoint

* Dirac notation: $|\psi\rangle \in \mathcal{H}$, $\langle\phi| \equiv |\phi\rangle^* \in \mathcal{H}^*$, $\langle\phi|\psi\rangle$ inner product

"ket" "bra" "bra-ket"

EX CLASS

$$\mathcal{H} = \mathbb{C}^d: \quad \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix} \qquad (\bar\phi_1 \cdots \bar\phi_d) \qquad \sum_i \bar\phi_i \psi_i$$

anti-linear

$\|\psi\| = 1$: $|\psi\rangle\langle\psi|$ = orthog. projection onto $\mathbb{C}|\psi\rangle$ CLEAR?

$\left[(|\psi\rangle\langle\psi|)^* = \langle\psi|^* \, |\psi\rangle^* = |\psi\rangle\langle\psi|, \ |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{=1}\langle\psi| = |\psi\rangle\langle\psi|, \ \operatorname{ran} = \mathbb{C}|\psi\rangle \right]$

Register: Σ finite set $\leadsto \mathcal{H} = \mathbb{C}^\Sigma$, ONB $\{|x\rangle\}_{x\in\Sigma}$ "standard basis"

e.g. qubit: $\Sigma = \{0,1\} \leadsto \mathcal{H} = \mathbb{C}^2$ w/ standard basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

bit

$$\boxed{\begin{array}{l} \text{A (q.) state on } X \text{ is an element of} \\ D(\mathcal{H}) = \{ \rho \in \operatorname{Pos}(\mathcal{H}), \ \operatorname{tr}[\rho] = 1 \} \end{array}}$$

positive semidefinite (PSD) operators

"density operator"

* ρ pure: $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathcal{H}$, $\|\psi\| = 1$
* maximally mixed state: $\tau = \frac{I}{d}$ $(d = \dim \mathcal{H})$

e.g.

* ρ classical on $\mathcal{H} = \mathbb{C}^\Sigma$: $\rho = \sum_x p_x |x\rangle\langle x|$

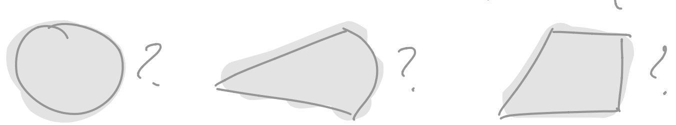
prob. dist.

$\rho = \begin{pmatrix} p_0 & 0 \\ 0 & p_1 \end{pmatrix}$ qubit

* $D(\mathcal{H})$ is convex: extreme points = pure states. Indeed:

SPECTRAL THM: $\forall \rho \ \exists$ ONB $\{|\psi_i\rangle\}$, prob. dist. $\{p_i\}$: $\boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$

What does $D(\mathcal{H})$ look like for a qubit?



Qubits: Pauli matrices $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}, X=\begin{pmatrix}0&1\\1&0\end{pmatrix}, Y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

are basis of 2×2 Herm. matrices (real vector space)

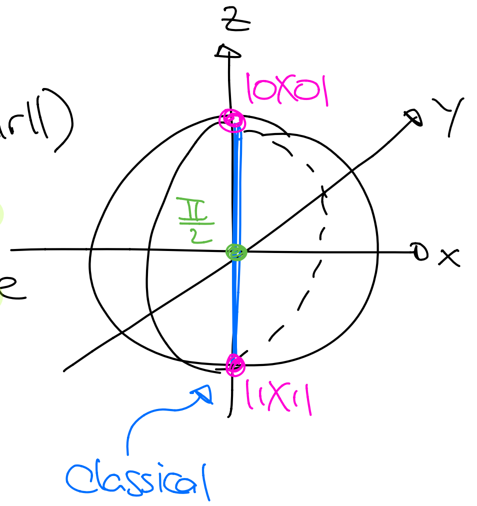
$\hookrightarrow \rho=\frac{1}{2}\begin{pmatrix}1+z & x-iy\\ x+iy & 1-z\end{pmatrix}$ Hermitian, $\mathrm{tr}=1$ $\quad r=\begin{pmatrix}x\\y\\z\end{pmatrix}$ Bloch vector

* eigenvalues: $\{p,1-p\}$ for $p\in\mathbb{R}$

  $\hookrightarrow p(1-p)=\det(\rho)=\frac{1}{4}(1-x^2-y^2-z^2)=\frac{1}{4}(1-\|r\|)$

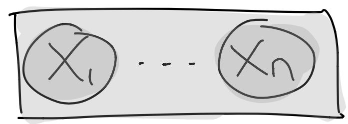
* ρ state: $p(1-p)\geq 0 \iff \|r\|\leq 1$ Bloch ball
* ρ pure: $p(1-p)=0 \iff \|r\|=1$ Bloch sphere
* ρ classical: $\rho=\begin{pmatrix}p & \\ & 1-p\end{pmatrix} \cong [0,1] \iff r=\begin{pmatrix}0\\0\\*\end{pmatrix}$

  Simplex



$\boxed{\text{EX. CLASS}}$

For a (q.) system composed of subsystems $X_1,\dots,X_n$, the Hilbert space is $\mathcal{H}=\mathcal{H}_1\otimes\dots\otimes\mathcal{H}_n$.



Register: $\Sigma=\Sigma_1\times\dots\times\Sigma_n$: $\mathcal{H}=\mathbb{C}^{\Sigma}\cong\mathbb{C}^{\Sigma_1}\otimes\dots\otimes\mathbb{C}^{\Sigma_n}$

w/ product basis $|x_1,\dots,x_n\rangle=|x_1\rangle\otimes\dots\otimes|x_n\rangle$

e.g. n qubits: $\Sigma=\{0,1\}^n$, $\mathcal{H}\cong(\mathbb{C}^2)^{\otimes n}$

* product state: $\rho=\rho_1\otimes\dots\otimes\rho_n$ where $\rho_i\in D(\mathcal{H}_i)$
* otherwise correlated $\quad$ e.g. $\frac{1}{2}(|0,0\rangle\langle 0,0|+|1,1\rangle\langle 1,1|)$

  (e.g. entanglement)

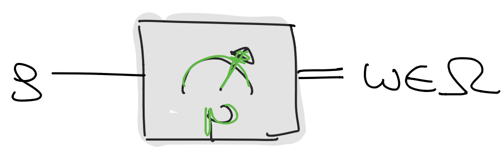
$$=\begin{pmatrix}\frac{1}{2}&&&\\&0&&\\&&0&\\&&&\frac{1}{2}\end{pmatrix}\in D(\mathbb{C}^2\otimes\mathbb{C}^2)$$

A measurement (POVM) on X with outcomes in Ω (finite set) is

$\mu:\Omega\to\mathrm{Pos}(\mathcal{H})$ s.th. $\sum_{\omega\in\Omega}\mu(\omega)=I_{\mathcal{H}}$

"POVM element"

Rule: If measure in state ρ, obtain outcome $\omega\in\Omega$ with probability $p(\omega)=\mathrm{tr}[\mu(\omega)\rho]$ prob. dist!



NB: Post-meas. state NOT specified (yet)!

* μ projective: $\mu(\omega)$ projections ($\forall \omega$) $\longrightarrow$ your QM class

* basis measurement: $\mu(\omega) = |\psi_\omega\rangle\langle\psi_\omega|$ for ONB $\{|\psi_\omega\rangle\}_{\omega\in\Omega}$ of $\mathcal{H}$

  e.g. standard basis of $\mathbb{C}^\Sigma \rightsquigarrow \mu: \Sigma \to \mathrm{Pos}(\mathcal{H})$

Qubits: Can measure in standard basis $\{|0\rangle, |1\rangle\}$, but also in Hadamard basis $\{|+\rangle, |-\rangle\}$, where $|\pm\rangle = \sqrt{\frac{1}{2}}(|0\rangle \pm |1\rangle)$

e.g. $\rho = \begin{pmatrix} 2/3 & \\ & 1/3 \end{pmatrix}$: $\left.\begin{array}{l} P_{std}(0) = \mathrm{tr}[|0\rangle\langle 0|\rho] = \langle 0|\rho|0\rangle = \frac{2}{3} \\ P_{Had}(+) = \mathrm{tr}[|+\rangle\langle +|\rho] = \langle +|\rho|+\rangle = \frac{1}{2} \end{array}\right\}$ CANNOT BOTH BE $=1$!

$\longrightarrow$ $\boxed{\text{EX CLASS}}$

Discriminating q. states:

WANT: Measurement $\mu: \{0,1\} \to \mathrm{Pos}(\mathcal{H})$ s.th.

$$P_{success} = \frac{1}{2}\mathrm{tr}[\mu(0)\rho_0] + \frac{1}{2}\mathrm{tr}[\mu(1)\rho_1] \text{ maximal!}$$

e.g. for a qubit:

* $\rho_0 = |0\rangle\langle 0|$, $\rho_1 = |1\rangle\langle 1|$: $P_{success} = 100\%$ via std. basis meas.
* $\rho_0 = \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix}$, $\rho_1 = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$: $P_{success} = \frac{2}{3}$ —— " —— optimal? In general?

In general: Set $\boxed{Q := \mu(0).}$ Then:

$$P_{success} = \frac{1}{2}\mathrm{tr}[Q\rho_0] + \frac{1}{2}\mathrm{tr}[(I-Q)\rho_1] = \frac{1}{2} + \frac{1}{2}\mathrm{tr}[Q(\rho_0 - \rho_1)]$$

$$\leq \frac{1}{2} + \frac{1}{4}\underbrace{\|\rho_0 - \rho_1\|_1}_{\text{trace distance}}$$

Here:

* $\|A\|_1 := \sum_i |a_i|$ trace norm of Hermitian A with eigenvalues $\{a_i\}$.
* for $A = \rho_0 - \rho_1 = \sum_i a_i |\psi_i\rangle\langle\psi_i|$:

$$\operatorname{tr}[QA] = \sum_i a_i \underbrace{\operatorname{tr}[Q|\psi_i\rangle\langle\psi_i|]}_{\in[0,1]} \leq \sum_{i:a_i>0} a_i = \frac{1}{2}\sum_i |a_i|$$

$\sum_i a_i = 0$

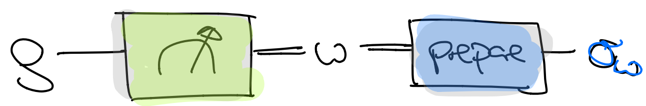
equality? $Q = \sum_{i:a_i>0} |\psi_i\rangle\langle\psi_i|$ (projection!)

$$\boxed{\begin{array}{l}\underline{\text{Thm}} \text{ (Helstrom): The optimal prob. of success is } \frac{1}{2} + \frac{1}{4}\|\rho_0 - \rho_1\|_1 \\ \text{It is achieved by a projective measurement.}\end{array}}$$

$\boxed{\text{PICTURES}}$

Q. Channels: $\rho_{in} \longrightarrow \boxed{???} \longrightarrow \rho_{out}$

* $\mathcal{I}_X[\rho] = \rho$ identity channel
* $\Phi[\rho] = U\rho U^\dagger$ for unitary U "base change"
* $\Phi[\rho] = \operatorname{tr}[\rho]\,\sigma$ for state σ discards input and prepares state σ
* $\Phi[\rho] = \rho \otimes \sigma$ for state σ prepare additional system in state σ
* "measure and prepare" channel:



home-work :)

$$\Phi[\rho] = \sum_\omega \operatorname{tr}[\mu(\omega)\rho]\cdot\sigma_\omega$$

where $\mu: \Omega \to \mathrm{Pos}(\mathcal{X})$ meas., $\{\sigma_\omega\}_{\omega\in\Omega}$ states on $\mathcal{Y}$

* depolarizing channel:

$$\Phi[\rho] = (1-p)\cdot\rho + p\cdot\frac{I}{d}\operatorname{tr}[\rho] \text{ for } p\in[0,1] \Big\} \text{ replace } \rho \text{ by } \frac{I}{d} \text{ w/ probability } p$$

General math. definition? In a later lecture, we will define the set

$C(\mathcal{X},\mathcal{Y})$

of (q) channels from system X to system Y.

For today + the homework, only important that:

① all examples above are indeed q. channels

② every q. channel $\Phi\in C(\mathcal{X},\mathcal{Y})$ is a linear map $\Phi: \underbrace{L(\mathcal{X})}_{\text{linear ops}} \to L(\mathcal{Y})$