Quantum Information Theory, Spring 2019

Exercise Set 8 **in-class practice problems**

1. **Warmup:**

   (a) Show that, if ρ and σ are both pure states, $D(\rho\|\sigma) \in \{0, \infty\}$.
   (b) Find a state ρ and a channel Φ such that $H(\Phi[\rho]) < H(\rho)$.
   (c) Compute the relative entropy $D(\rho\|\sigma)$ for $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ and $\sigma = \frac{1}{4}|+\rangle\langle +| + \frac{3}{4}|-\rangle\langle -|$.

2. **Matrix logarithm:** Recall that the logarithm of a positive definite operator with eigendecomposition $Q = \sum_i \lambda_i |e_i\rangle\langle e_i|$ is defined as $\log(Q) = \sum_i \log(\lambda_i)|e_i\rangle\langle e_i|$ (as always, our logarithms are to base 2). Verify the following properties:

   (a) $\log(cI) = \log(c)I$ for every $c \geq 0$.
   (b) $\log(Q \otimes R) = \log(Q) \otimes I_Y + I_X \otimes \log(R)$ for all positive definite operators Q on $\mathcal{X}$, R on $\mathcal{Y}$.
   (c) $\log(\sum_{x\in\Sigma} p_x |x\rangle\langle x| \otimes \rho_x) = \sum_{x\in\Sigma} \log(p_x)|x\rangle\langle x| \otimes I_Y + \sum_{x\in\Sigma} |x\rangle\langle x| \otimes \log(\rho_x)$ for every ensemble $\{p_x, \rho_x\}_{x\in\Sigma}$ of positive definite operators $\rho_x \in D(\mathcal{Y})$.

   Warning: It is in general not true that $\log(QR) = \log(Q) + \log(R)$*!*

3. **From Relative Entropy to Entropy and Mutual Information:** Use Problem 1 to verify the following claims from class:

   (a) $D(\rho\|\frac{I_X}{d}) = \log d - H(\rho)$ for every $\rho \in D(\mathcal{X})$, where $d = \dim \mathcal{X}$.
   (b) $D(\rho_{XY}\|\rho_X \otimes \rho_Y) = I(X:Y)_{\rho_{XY}}$ for every $\rho_{XY} \in D(\mathcal{X} \otimes \mathcal{Y})$, where $\rho_X = \mathrm{Tr}_Y[\rho_{XY}]$ and $\rho_Y = \mathrm{Tr}_X[\rho_{XY}]$. You may assume that all three operators are positive definite.

4. **Entropy and ensembles:** In this problem, you will prove the upper bound on the Holevo information that we discussed in class: For every ensemble $\{p_x, \rho_x\}$,

   $$\chi(\{p_x, \rho_x\}) \leq H(p) \quad \text{or, equivalently,} \quad H(\sum_x p_x \rho_x) \leq H(p) + \sum_x p_x H(\rho_x).$$

   Moreover, equality holds if and only if the ρ_x with $p_x > 0$ have pairwise orthogonal image.

   (a) First prove these claims assuming that each ρ_x is a pure state, i.e., $\rho_x = |\psi_x\rangle\langle\psi_x|$.
   Hint: Consider the pure state $|\Phi\rangle = \sum_x \sqrt{p_x}|x\rangle \otimes |\psi_x\rangle$ *and compare the entropy of the first system before and after measuring in the standard basis.*
   (b) Now prove the claims for general ρ_x.
   Hint: Apply part (a) to a suitable ensemble obtained from the eigendecompositions of the ρ_x*.*

   In terms of the cq-state corresponding to the ensemble, the above inequality can also be written as $H(XY) \geq H(Y)$. This confirms a claim made in Problem 2 of last week's exercise set.