Quantum Information Theory, Spring 2019

Exercise Set 3 **in-class practice problems**

1. **Positive, but not completely:** Show that the transpose superoperator $\Phi(X) = X^{\mathsf{T}}$ is positive but not completely positive.

2. **Quantum channels:** Recall that $P \in \mathrm{Pos}(\mathcal{X})$ if and only if $P = XX^*$, for some $X \in \mathrm{L}(\mathcal{X})$. Show that the following maps Φ are quantum channels by directly verifying that they are trace-preserving and completely positive.

   (a) (Basis change): Let $U \in \mathrm{U}(\mathcal{X})$ and define $\Phi(X) = UXU^*$, $\forall X \in \mathrm{L}(\mathcal{X})$.
   (b) (Adding a state): Let $\sigma \in \mathrm{D}(\mathcal{X})$ and define $\Phi(X) = \sigma \otimes X$, $\forall X \in \mathrm{L}(\mathcal{X})$.

3. **Across the channel:** Using the definitions of the natural representation $K(\Phi)$ and the Choi-Jamiołkowski representation $J(\Phi)$, verify the channel input-output relations.

   (a) (Natural representation): Verify that $K(\Phi)\,\mathrm{vec}(|a\rangle\langle b|) = \mathrm{vec}\big(\Phi(|a\rangle\langle b|)\big)$.
   (b) (Choi-Jamiołkowski representation): Verify that $\Phi(X) = \mathrm{Tr}_{\mathsf{X}}\big[J(\Phi) \cdot (I_{\mathsf{Y}} \otimes X^{\mathsf{T}})\big]$.

4. **Superoperator relations:**

   (a) (Kraus and Stinespring): Verify that the Kraus and Stinespring representations given in the class indeed correspond to the same superoperator:

   $$\Phi(X) = \sum_{a \in \Gamma} A_a X B_a^*, \qquad\qquad \Phi(X) = \mathrm{Tr}_{\mathsf{Z}}(AXB^*),$$

   where $A = \sum_{a \in \Gamma} A_a \otimes |a\rangle$ and $B = \sum_{b \in \Gamma} B_b \otimes |b\rangle$.
   (b) (Kraus and natural): Verify that the above Kraus representation is also equivalent to the natural representation

   $$K(\Phi) = \sum_{a \in \Gamma} A_a \otimes \bar{B}_a$$

   that treats operators as vectors: $K(\Phi) \cdot \mathrm{vec}(X) = \mathrm{vec}(\Phi(X))$.
   Hint: vectorize the Kraus representation.

5. **Unitary evolution:** Consider a quantum channel $\Phi(X) = UXU^*$ that corresponds to unitary evolution by some $U \in \mathrm{U}(\mathcal{X})$. Find its

   (a) natural representation $K(\Phi)$,
   (b) Choi representation $J(\Phi)$,
   (c) Kraus representation $\{\, A_a : a \in \Gamma \,\}$ and $\{\, B_a : a \in \Gamma \,\}$, for some set Γ,
   (d) Stinespring representation $A, B \in \mathrm{L}(\mathcal{X}, \mathcal{X} \otimes \mathcal{Z})$, for some space $\mathcal{Z} = \mathbb{C}^{\Gamma}$.

   What is the Choi representation $J(\Phi)$ of the identity channel $\Phi(X) = X$?

6. **Eaten up by trace:** Let $\mathcal{X} = \mathbb{C}^{\Sigma}$. Assume that $M \in \mathrm{L}(\mathcal{X})$ is such that $\mathrm{Tr}(MX) = \mathrm{Tr}(X)$, for all $X \in \mathrm{L}(\mathcal{X})$. Show that this implies $M = I_{\mathsf{X}}$.