Quantum Information Theory, Spring 2019

Exercise Set 15 **in-class practice problems**

1. **Self-adjoint map:** Verify that the map $\Phi \in \mathrm{T}(\mathcal{X} \oplus \mathcal{Y})$ given by

$$\Phi \begin{pmatrix} X & \cdot \\ \cdot & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \tag{1}$$

for all $X \in \mathrm{L}(\mathcal{X})$ and $Y \in \mathrm{L}(\mathcal{Y})$, is self-adjoint, i.e., $\Phi^* = \Phi$.

2. **SDP simplification:** Let $\Phi \in \mathrm{T}(\mathcal{X} \oplus \mathcal{Y})$ be the map given in Eq. (1). Let $K \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$ and define $A, B \in \mathrm{Herm}(\mathcal{X} \oplus \mathcal{Y})$ as follows:

$$A = \frac{1}{2} \begin{pmatrix} 0 & K^* \\ K & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} I_{\mathsf{X}} & 0 \\ 0 & I_{\mathsf{Y}} \end{pmatrix}.$$

Verify that the semidefinite program (Φ, A, B) is equivalent to the following:

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \mathrm{Re}(\langle K, Z \rangle) \\ \text{subject to:} \quad & \begin{pmatrix} I_{\mathsf{X}} & Z^* \\ Z & I_{\mathsf{Y}} \end{pmatrix} \geq 0, \\ & Z \in \mathrm{L}(\mathcal{X}, \mathcal{Y}). \end{aligned}$$

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \frac{1}{2}\mathrm{Tr}(X) + \frac{1}{2}\mathrm{Tr}(Y) \\ \text{subject to:} \quad & \begin{pmatrix} X & -K^* \\ -K & Y \end{pmatrix} \geq 0, \\ & X \in \mathrm{Pos}(\mathcal{X}), \\ & Y \in \mathrm{Pos}(\mathcal{Y}). \end{aligned}$$

3. **State discrimination SDP:** The goal of this exercise is to derive the dual for the SDP that describes the optimal measurement for discriminating quantum states from a given ensemble.

   (a) State the primal of the state discrimination problem

Primal problem

$$\begin{aligned} \text{maximize:} \quad & \sum_{a \in \Sigma} \langle \mu(a), \eta(a) \rangle \\ \text{subject to:} \quad & \mu(a) \in \mathrm{Pos}(\mathcal{X}), \ \forall a \in \Sigma, \\ & \sum_{a \in \Sigma} \mu(a) = I_{\mathsf{X}} \end{aligned}$$

in the standard form in terms of an appropriately chosen triple (Φ, A, B).

(b) Use this standard form to find the dual SDP. Write down the dual in the standard form.
(c) Show that the dual can be simplied to

Dual problem

$$\begin{aligned} \text{minimize:} \quad & \operatorname{Tr}(Y) \\ \text{subject to:} \quad & Y \geq \eta(a), \ \forall a \in \Sigma, \\ & Y \in \operatorname{Herm}(\mathcal{X}). \end{aligned}$$

4. **The pretty good measurement:** Recall that the *pretty good measurement* for an ensemble $\eta : \Sigma \to \operatorname{Pos}(\mathcal{X})$ is given by

$$\mu(a) = \rho^{-1/2}\eta(a)\rho^{-1/2}$$

where ρ is the average state:

$$\rho = \sum_{a \in \Sigma} \eta(a).$$

(You verified that this is indeed a valid measurement in Problem Set 4.)

(a) Let $|\Sigma| = d$ and assume that η corresponds to a uniform distribution over some orthonormal basis of $\mathbb{C}^d$, i.e., $\eta(a) = \frac{1}{d}|\psi_a\rangle\langle\psi_a|$ where $\langle\psi_a|\psi_b\rangle = \delta_{a,b}$. Show that for such ensemble the pretty good measurement is optimal.
(b) Show that the pretty good measurement is optimal for an orthonormal basis even if the probabilities for different states are arbitrary, i.e., $\eta(a) = p_a|\psi_a\rangle\langle\psi_a|$ for some probability distribution $(p_a : a \in \Sigma)$.