Quantum Information Theory, Spring 2019

Exercise Set 11 **in-class practice problems**

1. **Majorization examples:**
   (a) Let $p = (0.1, 0.7, 0.2)$ and $q = (0.3, 0.2, 0.5)$. Determine whether $p \prec q$ or $q \prec p$.
   (b) Find a sequence of Robin Hood transfers that converts one distribution into the other.
   (c) Express this sequence as a single stochastic matrix and verify that this matrix is in fact doubly stochastic.
   (d) Express this matrix as a convex combination of permutations.
   (e) Find a pair of probability distributions p and q such that neither $p \prec q$ nor $q \prec p$.

2. **Alternative definitions of majorization:** Let $u = (u_1, \dots, u_n)$ be a vector and let r denote *reverse sorting* and s denote *sorting*:

$$r_1(u) \geq r_2(u) \geq \cdots \geq r_n(u),$$
$$s_1(u) \leq s_2(u) \leq \cdots \leq s_n(u),$$

   such that $\{r_i(u) : i = 1, \dots, n\} = \{s_i(u) : i = 1, \dots, n\} = \{u_i : i = 1, \dots, n\}$ as multisets. Let v and u be two probability distributions over $\Sigma = \{1, \dots, n\}$, i.e., $v_i \geq 0$, $u_i \geq 0$, and $\sum_{i=1}^n v_i = \sum_{i=1}^n u_i = 1$. Show that the following conditions are equivalent:
   (a) $\sum_{i=1}^m r_i(u) \geq \sum_{i=1}^m r_i(v)$, for all $m \in \{1, \dots, n-1\}$.
   (b) $\sum_{i=1}^m s_i(u) \leq \sum_{i=1}^m s_i(v)$, for all $m \in \{1, \dots, n-1\}$.
   (c) $\forall t \in \mathbb{R} : \sum_{i=1}^n \max(u_i - t, 0) \geq \sum_{i=1}^n \max(v_i - t, 0)$.

3. **Vectorization and partial trace:**
   (a) Show that, for all $L, R \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$,

$$\mathrm{Tr}_{\mathsf{Y}}\big[\mathrm{vec}(L)\,\mathrm{vec}(R)^*\big] = LR^*.$$

   (b) Let $\Xi \in \mathrm{SepC}(\mathcal{X} : \mathcal{Y})$ be given by

$$\Xi(M) = \sum_{a \in \Sigma} (A_a \otimes B_a) M (A_a \otimes B_a)^*,$$

   for all $M \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$. Show that, for all $X \in \mathrm{L}(\mathcal{Y}, \mathcal{X})$,

$$\mathrm{Tr}_{\mathsf{Y}}\Big[\Xi\big(\mathrm{vec}(X)\,\mathrm{vec}(X)^*\big)\Big] = \sum_{a \in \Sigma} A_a X B_a^{\mathsf{T}} \bar{B}_a X^* A_a^*$$