Quantum Information Theory, Spring 2019

Exercise Set 1 **in-class practice problems**

1. **Dirac notation quiz:** In the Dirac notation, every vector is denoted by a ket (i.e., $|\cdot\rangle$) and every linear functional is denoted by a bra (i.e., $\langle\cdot|$). One can think of kets as column vectors and bras as row vectors. Hence, if $|\psi\rangle$ is a column vector, then $\langle\psi|$ denotes the row vector obtained by taking the conjugate transpose of the column vector.

   (a) Let $|\psi\rangle$ and $|\phi\rangle$ be vectors in $\mathbb{C}^n$ and A an $n \times n$ matrix. Which of the following expressions are syntactically correct? For those that do, what kind of object do they represent (e.g., numbers, vectors, ...)? Can you write them using 'ordinary' notation?

   i. $|\psi\rangle + \langle\phi|$
   ii. $|\psi\rangle\langle\phi|$
   iii. $A\langle\psi|$
   iv. $\langle\psi| A$
   v. $\langle\psi| A + \langle\psi|$
   vi. $|\psi\rangle\langle\phi| + A$
   vii. $|\psi\rangle\langle\phi| A$
   viii. $|\psi\rangle A \langle\phi|$
   ix. $\langle\psi| A |\phi\rangle$
   x. $\langle\psi| A |\phi\rangle + \langle\psi|\phi\rangle$
   xi. $\langle\psi|\phi\rangle\langle\psi|$
   xii. $\langle\psi|\phi\rangle A$

   (b) Let $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ be two pure states on the same system. Verify that

   $$\mathrm{tr}[\rho\sigma] = |\langle\psi|\phi\rangle|^2.$$

   Hint: You may use that the trace is cyclic, i.e. $\mathrm{tr}[ABC] = \mathrm{tr}[CAB] = \mathrm{tr}[BCA]$.

2. **Bloch sphere bonanza:** Recall from the lecture that one can visualize any one-qubit state ρ by its Bloch vector $\vec{r} \in \mathbb{R}^3$, $\|\vec{r}\| \leq 1$.

   (a) Let σ be another qubit state, with Bloch vector $\vec{s}$. Verify that

   $$\mathrm{tr}[\rho\sigma] = \frac{1}{2}\left(1 + \vec{r} \cdot \vec{s}\right).$$

   (b) Let $\{|\psi_x\rangle\}_{x=0,1}$ denote an orthonormal basis of $\mathbb{C}^2$, $\mu\colon \{0,1\} \to \mathrm{Pos}(\mathbb{C}^2)$ the corresponding basis measurement (i.e., $\mu(x) = |\psi_x\rangle\langle\psi_x|$ for $x \in \{0,1\}$), and $\vec{r}_x$ the Bloch vector of $|\psi_x\rangle\langle\psi_x|$ for $x \in \{0,1\}$. Show that $\vec{r}_0 = -\vec{r}_1$. Moreover, show that the probability of obtaining outcome $x \in \{0,1\}$ when measuring ρ using μ is given by $\frac{1}{2}(1 + \vec{r} \cdot \vec{r}_x)$. How can you visualize these two facts on the Bloch sphere?

   (c) Now imagine that ρ is an unknown qubit state ρ whose Bloch vector $\vec{r}$ you would like to characterize completely. Consider the following measurement with six outcomes:

   $$\mu\colon \{x,y,z\} \times \{0,1\} \to \mathrm{Pos}(\mathbb{C}^2), \quad \mu(a,b) = \frac{I + (-1)^b \sigma_a}{6},$$

   where $\sigma_x = X$, $\sigma_y = Y$, and $\sigma_z = Z$ are the three Pauli matrices. Show that μ is a valid measurement and that the probabilities of measurement outcomes are given by

   $$p(a,b) = \frac{1 + (-1)^b r_a}{6}.$$

   How can you visualize this formula on the Bloch sphere? Describe how measuring many copies of ρ by using μ allows for estimating the entries of $\vec{r}$ to arbitrary accuracy.