Symmetry and Quantum Information March 19, 2018

Problem Set 6

Michael Walter, University of Amsterdam *due March 27, 2018*

Problem 1 (Gentle measurement, 3 points).
In this problem, you will derive a useful technical result known as the *gentle measurement lemma.* Let ρ be a quantum state and $0 \leq Q \leq I$ a POVM element.

(a) Show that if $\operatorname{tr}[\rho Q] \geq 1 - \varepsilon$ then $T(\rho, \frac{\sqrt{Q}\rho\sqrt{Q}}{\operatorname{tr}[\rho Q]}) \leq \sqrt{\varepsilon}$.

Hint: First prove the result for pure states.

(b) Explain in one sentence why this result is called the *gentle measurement lemma.*

Problem 2 (Quantum data compression, 3 points).
In this problem you will show that there cannot exist typical subspaces with rates smaller than the von Neumann entropy. Thus, let ρ be a density operator on $\mathbb{C}^2$ and $\mathcal{H}_n \subseteq (\mathbb{C}^2)^{\otimes n}$ an arbitrary sequence of subspaces, with corresponding projectors P_n, such that $\dim(\mathcal{H}_n) \leq 2^{nR}$ for all n. Show that either $R \geq S(\rho)$ or $\operatorname{tr}[\rho^{\otimes n} P_n] \to 0$.

Problem 3 (Schur-Weyl duality, 8 points).
Your goal in this exercise is to concretely identify irreducible representations of $U(2)$ and of S_n in the n-qubit Hilbert space, and to explicitly realize the Schur-Weyl duality in a special case. Let $k \in \{0, 1, \ldots, n\}$ be an integer such that $n - k$ is even.

(a) Show that the invariant subspace

$$V'_{n,k} \coloneqq \left\{ |\phi\rangle \otimes |\Psi^-\rangle^{\otimes (n-k)/2} : \ |\phi\rangle \in \operatorname{Sym}^k(\mathbb{C}^2) \right\} \subseteq (\mathbb{C}^2)^{\otimes n}$$

is an irreducible $U(2)$-representation equivalent to $V_{n,k}$. As always, $|\Psi^-\rangle$ denotes the singlet, and $U(2)$ acts on $(\mathbb{C}^2)^{\otimes n}$ by $U^{\otimes n}$. How can you obtain further $U(2)$-representations in $(\mathbb{C}^2)^{\otimes n}$ that are equivalent to $V_{n,k}$?

Hint: Recall the symmetry of the singlet state from Problem Set 2.

(b) Show that the invariant subspace

$$W'_{n,k} \coloneqq \operatorname{span}\left\{ R_\pi \left(|0\rangle^{\otimes k} \otimes |\Psi^-\rangle^{\otimes (n-k)/2} \right) : \pi \in S_n \right\} \subseteq (\mathbb{C}^2)^{\otimes n}$$

is an irreducible S_n-representation equivalent to $W_{n,k}$. How can you obtain further S_n-representations in $(\mathbb{C}^2)^{\otimes n}$ equivalent to $W_{n,k}$?

Hint: You are allowed to use the statement of Schur-Weyl duality.

Now consider the case of three qubits. Here, $n = 3$, so the only two options for k are $k = 1, 3$.

(c) Show that $W_{3,3}$ is equivalent to the trivial representation $\mathbb{C}$, while $W_{3,1}$ is equivalent to the two-dimensional irreducible representation $\mathcal{H} = \{(\alpha, \beta, \gamma) : \alpha + \beta + \gamma = 0\}$ from Problem 3.1.

(d) Construct a unitary operator $(V_{3,3} \otimes \mathbb{C}) \oplus (V_{3,1} \otimes \mathcal{H}) \to (\mathbb{C}^2)^{\otimes 3}$ that is an intertwiner for the actions of both $U(2)$ and S_3.

Hint: In (c), construct an explicit intertwiner $\mathcal{H} \cong W'_{3,1}$ that you can re-use in (d).