Symmetry and Quantum Information March 12, 2018

Problem Set 5

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Problem 1 (Monotonicity of the trace distance, 1 point)**.**
Show that, for every two density operators ρ_{AB} and σ_{AB}, $T(\rho_A, \sigma_A) \leq T(\rho_{AB}, \sigma_{AB})$.

Problem 2 (Purifications, 5 points)**.**
In this problem, you will establish some useful facts concerning purifications that will also be helpful in the remainder of this problem set. Throughout, let ρ_A be a density operator on a Hilbert space $\mathcal{H}_A$. First, you will show that any two purifications are related by an isometry:

(a) Show that if $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\Phi_{AC}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_C$ are two purifications of ρ_A such that $\dim \mathcal{H}_B \leq \dim \mathcal{H}_C$, then there exists an isometry $V_{B\to C}$ such that $|\Phi_{AC}\rangle = (I_A \otimes V_{B\to C}) |\Psi_{AB}\rangle$.

Hint: Use the Schmidt decomposition.

In particular, when $\mathcal{H}_B \cong \mathcal{H}_C$ then this shows that the two purifications are related by a unitary, which is something we asserted but did not prove in class.

Next, you will construct a particular purification of ρ_A (sometimes called the *standard purification*) and see how symmetries can be lifted. For simplicity, assume that $\mathcal{H}_A = \mathbb{C}^d$.

(b) Show that $|\Psi_{AB}\rangle := (\sqrt{\rho_A} \otimes I_B) \sum_{i=1}^d |ii\rangle$ is always a purification of ρ_A. Here, $\mathcal{H}_B = \mathbb{C}^d$, and $\sqrt{\rho_A}$ is defined by taking the square root of each eigenvalue of ρ_A while keeping the same eigenspaces.

(c) Show that this purification has the following property: For every unitary U_A, $[U_A, \rho_A] = 0$ implies that $(U_A \otimes \bar{U}_B) |\Psi_{AB}\rangle = |\Psi_{AB}\rangle$. Here, $\bar{U}_B$ denotes the complex conjugate of U_A.

Problem 3 (De Finetti theorem for permutation-invariant quantum states, 5 points)**.**
In this problem, you will extend the quantum de Finetti theorem from states on the symmetric subspace to arbitrary permutation-invariant states. A quantum state $\rho_{A_1\dots A_N}$ is called *permutation-invariant* if $[R_\pi, \rho_{A_1\dots A_N}] = 0$ for all $\pi \in S_N$.

(a) Give two examples of permutation-invariant quantum states that are not just states on the symmetric subspace.

Now let $\rho_{A_1\dots A_N}$ be an arbitrary permutation-invariant quantum state on $(\mathbb{C}^d)^{\otimes N}$.

(b) Show that the reduced density operators for any fixed number of subsystems are all the same. That is, show that $\rho_{A_{i_1}\dots A_{i_k}} = \rho_{A_1\dots A_k}$ for all $1 \leq k \leq N$ and pairwise distinct indices $i_1, \dots, i_k$.

By monogomy, we would therefore expect that a de Finetti theorem should also hold in this situation. You will prove this in the remainder of this exercise:

(c) Show that there exists a pure state $\rho_{(A_1B_1)\dots(A_NB_N)}$ on $\mathrm{Sym}^N(\mathbb{C}^d \otimes \mathbb{C}^d) \subseteq (\mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes N}$ such that $\rho_{A_1\dots A_N} = \mathrm{tr}_{B_1\dots B_N}[\rho_{(A_1B_1)\dots(A_NB_N)}]$.

(d) Conclude that, for every $1 \leq k \leq N$, there exists a probability measure $d\mu$ on the set of density operators on $\mathbb{C}^d$ such that $T(\rho_{A_1\dots A_k}, \int d\mu(\rho)\, \rho^{\otimes k}) \leq \sqrt{d^2 k/n}$, where $n = N - k$.

Problem 4 (Universal classical data compression, 4 points)**.**
Given $R > 0$, construct a data compression protocol at asymptotic rate R that works for every classical data source that emits bits with probabilities $\{p, 1-p\}$ such that $h(p) < R$.