Symmetry and Quantum Information	March 5, 2018
Problem Set 4	
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Problem 1 (Pure state entanglement, 3 points).
In class we observed that a pure state $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is *unentangled* if and only if its reduced density operators ρ_A and ρ_B are pure states. Here you will generalize this observation and show that the maximal fidelity squared between $|\Psi_{AB}\rangle$ and any product state is given by the largest eigenvalue of ρ_A, denoted $\lambda_{\max}(\rho_A)$. That is, show that

$$\max_{\|\phi_A\|=\|\psi_B\|=1} |\langle\Psi_{AB}|\phi_A \otimes \psi_B\rangle|^2 = \lambda_{\max}(\rho_A).$$

Hint: Use the Schmidt decomposition discussed in Lecture 8.

Problem 2 (De Finetti and mean field theory, 4 points).
In this exercise you will explore the consequences of the quantum de Finetti theorem for mean field theory. Consider a Hermitian operator h on $\mathbb{C}^d \otimes \mathbb{C}^d$ and the corresponding *mean-field Hamiltonian*, i.e., the operator

$$H = \frac{1}{n-1} \sum_{i \neq j} h_{i,j}$$

on $(\mathbb{C}^d)^{\otimes n}$, where each term $h_{i,j}$ acts by the operator h on subsystems i and j and by the identity operator on the remaining subsystems (e.g., $h_{1,2} = h \otimes I^{\otimes(n-2)}$).

(a) Show that the eigenspaces of H are invariant subspaces for the action of the symmetric group.

Now assume that the eigenspace with minimal eigenvalue (the so-called *ground space*) is nondegenerate and spanned by some $|E_0\rangle$, with corresponding eigenvalue E_0. Then part (a) implies that $R_\pi |E_0\rangle = \chi(\pi) |E_0\rangle$ for some function χ. This function necessarily satisfies $\chi(\pi\tau) = \chi(\pi)\chi(\tau)$.

(b) Show that $\chi(i \leftrightarrow j) = \chi(1 \leftrightarrow 2)$ for all $i \neq j$. Conclude that $|E_0\rangle$ is either a symmetric tensor or an antisymmetric tensor.

   Hint: First show that $\chi(\pi\tau\pi^{-1}) = \chi(\tau)$.

If $n > d$, then there exist no nonzero antisymmetric tensors. Thus, in the so-called *thermodynamic limit* of large n, the ground state $|E_0\rangle$ is in the symmetric subspace $\mathrm{Sym}^n(\mathbb{C}^d)$ and so the quantum de Finetti theorem is applicable.

(c) Show that, for large n, the energy density in the ground state can be well approximated by minimizing over tensor power states. That is, show that

$$\frac{E_0}{n} \approx \min_{|\psi\rangle} \langle\psi^{\otimes 2}|h|\psi^{\otimes 2}\rangle = \frac{1}{n} \min_{|\psi\rangle} \langle\psi^{\otimes n}|H|\psi^{\otimes n}\rangle .$$

   Hint: The following fact about the trace distance will be useful. If ρ, σ *are density operators and* O *an observable, then* $|\mathrm{tr}[O\rho] - \mathrm{tr}[O\sigma]| \leq 2\|O\|_\infty T(\rho,\sigma)$, *where* $\|O\|_\infty := \max_{\|\phi\|=1} |\langle\phi|O|\phi\rangle|$.

This justifies the folklore that "in the mean field limit the ground state has the form $|\psi\rangle^{\otimes\infty}$".

Problem 3 (The antisymmetric state, 5 points).
In class, we discussed the quantum de Finetti theorem for the symmetric subspace. It asserts that the reduced density operators $\rho_{A_1\ldots A_k}$ of a state on $\mathrm{Sym}^{k+n}(\mathbb{C}^D)$ are $\sqrt{kD/n}$ close in trace distance to a separable state (in fact, to a mixture of tensor power states).

The goal of this exercise is to show that some kind of dependence on the dimension D is unavoidable in the statement of the theorem. To start, consider the *Slater determinant*

$$|S\rangle_{A_1\ldots A_d} = |1\rangle \wedge \cdots \wedge |d\rangle := \sqrt{\frac{1}{d!}} \sum_{\pi\in S_d} \mathrm{sign}(\pi)\, |\pi(1)\rangle \otimes \ldots \otimes |\pi(d)\rangle \in (\mathbb{C}^d)^{\otimes d}.$$

We define the *antisymmetric state* on $\mathbb{C}^d \otimes \mathbb{C}^d$ by tracing out all but two subsystems,

$$\rho_{A_1A_2} = \mathrm{tr}_{A_3\ldots A_d}\left[|S\rangle\langle S|\right].$$

(a) Let $F = R_{1\leftrightarrow 2}$ denote the swap operator on $(\mathbb{C}^d)^{\otimes 2}$. Prove the following identity, which is known as the *swap trick*:

$$\mathrm{tr}[F(\sigma\otimes\gamma)] = \mathrm{tr}[\sigma\gamma]$$

(b) Show that $T(\rho_{A_1A_2}, \sigma_{A_1A_2}) \geq \frac{1}{2}$ for all separable states $\sigma_{A_1A_2}$.

Hint: Consider the POVM element $Q = \Pi_2$ (i.e., the projector onto the symmetric subspace).

Thus you have shown that the antisymmetric state is far from any separable state. However, note that $|S\rangle$ is *not* in the symmetric subspace.

(c) Show that $|S\rangle^{\otimes 2} \in \mathrm{Sym}^d(\mathbb{C}^d\otimes\mathbb{C}^d)$, while $\rho^{\otimes 2}_{A_1A_2}$ is likewise far away from any separable state. Conclude that the quantum de Finetti theorem must have some dimension dependence.

Hint: $|S\rangle^{\otimes 2}$ is a state of $2d$ quantum systems that we might label $A_1\ldots A_d A'_1\ldots A'_d$ (the unprimed systems refer to the first copy of $|S\rangle$ and the primed to the second). Let the permutation group S_d act by simultaneously permuting unprimed and primed systems and show that $|S\rangle^{\otimes 2}$ is in the corresponding symmetric subspace. Similarly, $\rho^{\otimes 2}$ is an operator on $A_1A_2A'_1A'_2$. How do you need to partition the systems so that $\rho^{\otimes 2}$ is far from being separable?

Problem 4 (Classical data compression, 4 points).
In this exercise you will show that the Shannon entropy $h(p) = -p\log p - (1-p)\log(1-p)$ is the optimal compression rate for the coin flip problem discussed in class. Assume that Alice compresses her random sequence of n coin flips by applying a function $\mathcal{E}_n\colon \{H,T\}^n \to \{0,1\}^{\lfloor nR\rfloor}$, and Bob decompresses by applying a corresponding function $\mathcal{D}_n\colon \{0,1\}^{\lfloor nR\rfloor} \to \{H,T\}^n$.

(a) Which are the coin flip sequences that are transmitted correctly? Find an upper bound on their cardinality in terms of R.

(b) Show that, if $R < h(p)$, then the probability of success tends to zero for large n.

Hint: Distinguish between typical and atypical sequences of coin flips.

The following exercises are offered as additional opportunity for practice. They will not be graded.

Optional Problem 5 (Entanglement witness for the ebit)**.**
Recall that an *entanglement witness* for a quantum state ρ_{AB} is an observable O_{AB} such that $\operatorname{tr}[O_{AB}\,\rho_{AB}] > 0$, while $\operatorname{tr}[O_{AB}\,\sigma_{AB}] \leq 0$ for every separable state σ_{AB}. Construct an entanglement witness for the ebit state $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Hint: Use the claim of Problem 1 to your advantage!

Optional Problem 6 (Trace distance and observables)**.** In this problem, you will show that density operators ρ and σ with small trace distance $T(\rho, \sigma)$ have similar expectation values.

(a) Show that, for every two Hermitian operators M and N, $|\operatorname{tr}[MN]| \leq \|M\|_1 \|N\|_\infty$. Here, $\|M\|_1$ is the *trace norm* that you know from class (i.e., the sum of absolute values of the eigenvalues of M) and $\|N\|_\infty := \max_{\|\phi\|=1} |\langle\phi|N|\phi\rangle|$ is the *operator norm* (which can also be defined as the maximal absolute value of the eigenvalues of N).

(b) Conclude that, for every observable O, $|\operatorname{tr}[\rho O] - \operatorname{tr}[\sigma O]| \leq 2\,\|O\|_\infty\, T(\rho, \sigma)$.

This confirms the hint given in Problem 2, part (c).

(c) Find a (nonzero) observable for which the bound in part (b) is an equality.