Last time: Q. info source: $\{p_i, |\psi_i\rangle\} \rightsquigarrow \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Classical & quantum data compression

A classical tale: Alice aquired a biased coin:

$$\text{ALICE} \xrightarrow{\text{How many bits?}} \text{BOB}$$

$p = 75\%$

$1-p = 25\%$

1 bit otherwise: 25% error

n coin flips? Can we do better than $\boxed{1 \dfrac{\text{bit}}{\text{coin flip}}}$?

* Typical sequences have $\boxed{\dfrac{k}{n} \approx p}$ Compression rate

$$\overbrace{\underbrace{HTTHTH \cdots HTH}_{k \text{ heads},\ n-k \text{ tails}}}^{n}$$ random sequence

Isn't H...H more likely? Yes but...

LAW OF LARGE NUMBERS: $\forall \varepsilon > 0$

$$\Pr\left(\left|\frac{k}{n} - p\right| > \varepsilon\right) = O\left(\frac{1}{n}\right) \longrightarrow 0 \quad \text{as } n \to \infty$$

* NB: Good way to estimate p!

* How many sequences with k heads? $\binom{n}{k}$

Asymptotics: For all $x \in [0,1]$:

$$1 = (x + (1-x))^n = \sum_{\ell=0}^{n} \binom{n}{\ell} x^\ell (1-x)^{n-\ell} \geq \binom{n}{k} x^k (1-x)^{n-k}$$

$x = \frac{k}{n}$

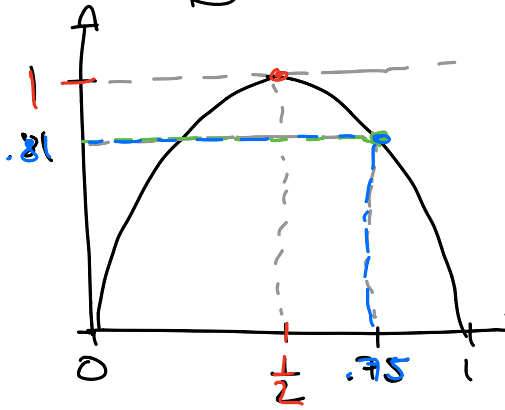
$$\Longrightarrow \binom{n}{k} \le x^{-k}(1-x)^{-(n-k)} = \left(\frac{k}{n}\right)^{-k}\left(1-\frac{k}{n}\right)^{-(n-k)}$$

$$\Longrightarrow \frac{1}{n}\log\binom{n}{k} \le -\frac{k}{n}\log\frac{k}{n} - \left(1-\frac{k}{n}\right)\log\left(1-\frac{k}{n}\right) =: h\left(\frac{k}{n}\right)$$

where we defined the binary Shannon entropy

$$h(p) := -p\cdot\log(p) - (1-p)\log(1-p)$$

BASE 2



* $2^{n\cdot h\left(\frac{k}{n}\right)}$ sequences

↳ need only $n\cdot h\left(\frac{k}{n}\right) = n\cdot\left(h(p\pm\varepsilon)\right) = n\left(h(p)\pm\varepsilon'\right)$ bits

* $p = 75\% \Longrightarrow h(p) \approx 81\%$

$p = 50\% \Longrightarrow h(p) = 100\%$ } Intuitive?

Compression protocol: Fix $\varepsilon > 0$.

* If $\left|\frac{k}{n} - p\right| > \varepsilon$: FAIL (send over arbitrary message)

* Send k to Bob. log(n) bits

* Send index of sequence in list of all coin flip sequences with k heads } $n\cdot(h(p)+\varepsilon')$ bits

Analysis: * $\Pr(\text{FAIL}) \longrightarrow 0$ as $n \to \infty$ law of large nos

* Rate: $R = \underbrace{\cancel{\frac{\log(n)}{n}}}_{\to 0} + \boxed{h(p)} + \underbrace{\boxed{\varepsilon'}}_{\text{as small as we want}} \ \frac{\text{bits}}{\text{coin flip}}$ ☺

* $\underline{\text{Entropy}}$ $h(p)$ is $\underline{\text{optimal}}$ asymptotic rate $\boxed{\text{PSET}}$

* Symmetries? Ask
  - $\Pr(\text{sequence}) = p^k (1-p)^{n-k}$ permutation symmetry
  - $h(p) = h(1-p)$:-) relabeling should not improve rate

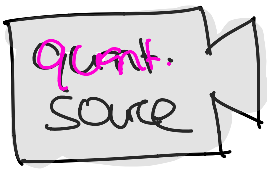
  :-(need to change protocol universal compressor?

* Instead of failing, can also send uncompressed string!

  ↳ expected rate $\approx \boxed{h(p)}$ because $\Pr(\text{fail}) \to 0$

Quantum compression

What's the model?

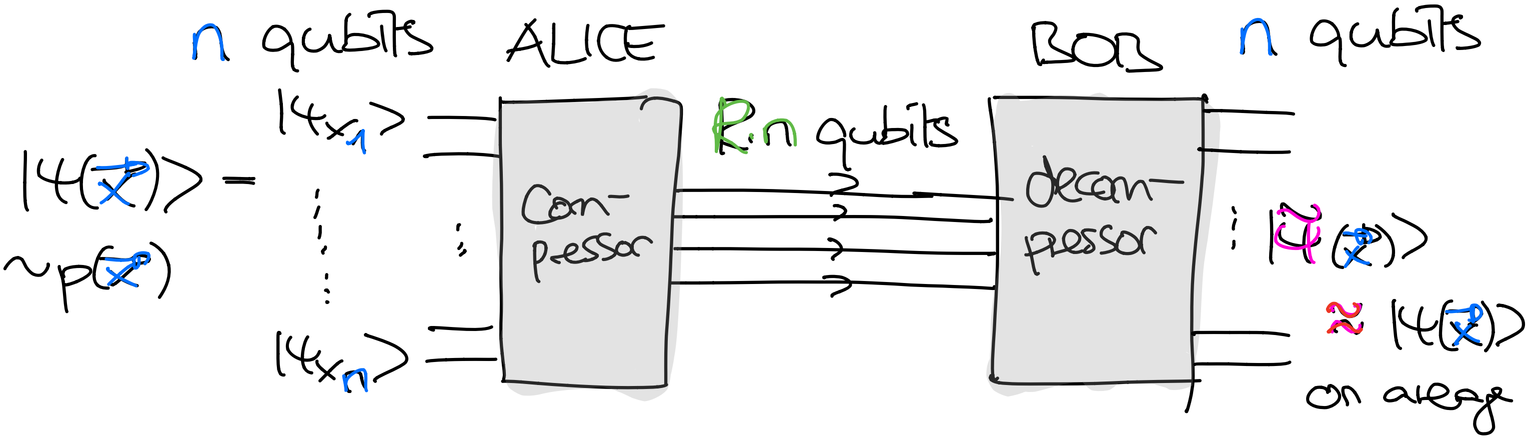


emits $|\psi_x\rangle$ with prob p_x

↳ emits $|\psi(\vec{x})\rangle = |\psi_{x_1}\rangle \otimes \cdots \otimes |\psi_{x_n}\rangle$

with prob. $p(\vec{x}) = p_{x_1} \cdot \ldots \cdot p_{x_n}$ $\Bigg\}$ $\hat{=}$ n coin tosses

Goal: Design compressor & decompressor s.th.



Formula?

$$\boxed{\sum_{\vec{x}} p(\vec{x}) \cdot \mathbb{E}\left[|\langle \tilde{\psi}(\vec{x}) | \psi(\vec{x}) \rangle|^2 \right] \overset{!}{\approx} 1}$$

Compression might invoke random measurement outcomes

<u>Average output of source:</u>

$\rho = \sum_x p_x |\psi_x\rangle\langle\psi_x| \rightsquigarrow$ eigenvalues $\{p, 1-p\}$

* n samples $\rightsquigarrow \sum_{\vec{x}} p(\vec{x}) |\psi(\vec{x})\rangle\langle\psi(\vec{x})| = \rho^{\otimes n}$ $\boxed{\text{quantum IID}}$
* many ensembles can give same ρ
* $|\psi_x\rangle$ need NOT be orthogonal $\Rightarrow \{p_x\} \neq \{p, 1-p\}$ $\lightning$

CANNOT just measure to figure out $\vec{x}$

e.g. $\frac{1}{2}(|0\rangle\langle 0| + |+\rangle\langle +|) = \rho$ has $p \approx 85\%$

<u>Idea:</u> Find "small" subspace $H_n \subseteq (\mathbb{C}^2)^{\otimes n}$ that contains "typical" $|\psi(\vec{x})\rangle$. $\leftarrow$ quantum version of testing if $|\frac{k}{n} - p| \leq \varepsilon$

<u>Compression protocol:</u> Let $P_n :=$ projector onto H_n.

Measure $\{P_n, I - P_n\}$.

typical: Post-measurement state

$$|\tilde{\psi}(\vec{x})\rangle = \frac{P_n |\psi(\vec{x})\rangle}{\| P_n |\psi(\vec{x})\rangle \|} \in H_n$$

atypical: FAIL, e.g., send over arbitrary state $|\tilde{\psi}(\vec{x})\rangle$

$\Downarrow$

Alice can send $|\tilde{\psi}(\vec{x})\rangle$ to Bob using only $\log(\dim \mathcal{H}_n)$ qubits

How? Set $\ell := \log(\dim \mathcal{H}_n)$. Alice applies

$$(\mathbb{C}^2)^{\otimes n} \xrightarrow{U} (\mathbb{C}^2)^{\otimes \ell} \otimes (\mathbb{C}^2)^{\otimes (n-\ell)}$$

$$\mathcal{H}_n \hookrightarrow (\mathbb{C}^2)^{\otimes \ell} \otimes |0\ldots0\rangle$$

& sends over first ℓ qubits. Bob adds $|0\ldots0\rangle$ & does $U^\dagger$.

<u>Analysis</u>: * Rate: $\dfrac{\log \dim(\mathcal{H}_n)}{n} \leq 1.$

* $\Pr_{\psi(\vec{x})}(\text{typical}) = \langle \psi(\vec{x}) | P_n | \psi(\vec{x}) \rangle =: q(\vec{x})$

* Average square fidelity?

$$\sum_{\vec{x}} p(\vec{x})\, E\left[|\langle \psi(\vec{x}) | \tilde{\psi}(\vec{x}) \rangle|^2 \right]$$

$$= \sum_{\vec{x}} p(\vec{x}) \cdot \left\{ q(\vec{x})\, \left| \langle \psi(\vec{x}) | \frac{P_n |\psi(\vec{x})\rangle}{\underbrace{\| P_n |\psi(\vec{x})\rangle \|}_{\sqrt{q(\vec{x})}}} \rangle \right|^2 + \underbrace{\boxed{\ldots}}_{\geq 0} \right\}$$

$$= \sum_{\vec{x}} p(\vec{x}) \cdot q^2(\vec{x}) \overset{\text{JENSEN}}{\geq} \left(\sum_{\vec{x}} p(\vec{x}) \cdot q(\vec{x}) \right)^2$$

where

$$\sum_{\vec{x}} p(\vec{x}) \cdot q(\vec{x}) = \sum_{\vec{x}} p(\vec{x}) \cdot \operatorname{tr}\left[|\psi(\vec{x})\rangle\langle\psi(\vec{x})| \cdot P_n \right] \underset{\text{SEE ABOVE}}{=} \operatorname{tr}\left[\rho^{\otimes n} P_n \right]$$

Thus: Q. compression at rate R is possible if

$$\boxed{\begin{array}{ll} ① \ \frac{1}{n} \log \dim(\mathcal{H}_n) \leq R & \\ & \text{for large } n \\ ② \ \operatorname{tr}[\rho^{\otimes n} P_n] \approx 1 & \end{array}}$$

$\mathcal{H}_n$ is called typical subspace

NB: Only involves ρ, not details of q. source!

↳ protocol works for all sources w/ $\sum_x p_x |\psi_x\rangle\langle\psi_x| = \rho$

Tomorrow:

* Construction of typical subspace using ideas from classical data compression
* Relationship between compression & entanglement