Last week: * Intro to representation theory

* $\mathrm{Sym}^k(\mathbb{C}^2)$ is irreducible (S)U(2)-rep.

$R_U^{(k)} :=$ restriction of $U^{\otimes k}$ to $\mathrm{Sym}^k(\mathbb{C}^2)$

Today, we will see that we achieved much more!!!

Representation Theory of SU(2):

IMPORTANT

* Any irreducible (unitary) representation of SU(2) is equivalent to some $\mathrm{Sym}^k(\mathbb{C}^2)$:

i.e. if $\mathcal{H}$ is irreducible with repr. operators $\{R_U\}$:

$\exists k$ & (unitary) intertwiner $J: \mathcal{H} \to \mathrm{Sym}^k(\mathbb{C}^2)$:

$$J R_U J^\dagger = R_U^{(k)} \qquad \forall U \in SU(2)$$

NOTATION: $\mathcal{H} \cong \mathrm{Sym}^k(\mathbb{C}^2)$, $R_g \cong R_g^{(k)}$

* For $k \neq l$, inequivalent: $\mathrm{Sym}^k(\mathbb{C}^2) \not\cong \mathrm{Sym}^l(\mathbb{C}^2)$

* How to determine k? $\dim \mathrm{Sym}^k(\mathbb{C}^2) = k+1$

Only useful if we already know that an irrep

* PHYSICS: $\mathrm{Sym}^k(\mathbb{C}^2)$ is representation w/ spin $j = \frac{k}{2}$

Examples:

* $\mathbb{C}^2 = \mathrm{Sym}^1(\mathbb{C}^2)$ spin $\frac{1}{2}$

* $\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2}_{\text{representation of } SU(2)} = \underbrace{\mathrm{Sym}^2(\mathbb{C}^2)}_{3\text{-dim}} \overset{\perp}{\oplus} \underbrace{\mathbb{C}|\Psi^-\rangle}_{\substack{1\text{-dim} \\ \text{so must be} \cong \mathrm{Sym}^0(\mathbb{C}^2)}}$ PSET 3

where $|\Psi^-\rangle = \sqrt{\frac{1}{2}}(|10\rangle - |01\rangle)$ is the singlet.

For all $U \in SU(2)$:

$$(U \otimes U)|\Psi^-\rangle \overset{\text{PSET 2}}{=} \det(U)|\Psi^-\rangle = |\Psi^-\rangle$$ trivial repr!

$$\Rightarrow \boxed{\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathrm{Sym}^2(\mathbb{C}^2) \oplus \mathrm{Sym}^0(\mathbb{C}^2)}$$

Spin: $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

How to do this more systematically?

How to decompose a given representation $\mathcal{H}$, with operators $\{R_U\}_{U \in SU(2)}$?

* As before, define $r_Z := -i\,\partial_{t=0}\big[R_{\boxed{e^{itZ}}}\big]$, $e^{itZ} \in SU(2)$

* If $\mathcal{H} \cong \mathrm{Sym}^k(\mathbb{C}^2)$: Eigenvalues of r_Z are $\{k, k-2, \dots, -k\}$

Pf: $R_{e^{itZ}} \cong R^{(k)}_{e^{itZ}} \Longrightarrow r_Z \cong r_Z^{(k)}$ w/ eigenvectors $|\omega_{m,k-m}\rangle$ eigenvalues $(2m-k)$ $m \in \{0,1,\dots,k\}$ $\square$

* In general:

$$\mathcal{H} \cong \mathrm{Sym}^{k_1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{k_2}(\mathbb{C}^2) \oplus \ldots \oplus \mathrm{Sym}^{k_m}(\mathbb{C}^2)$$

each contributes $\{-k_i, \ldots, k_i-2, k_i\}$

$\hookrightarrow$ can reverse-engineer $k_1, \ldots, k_m$ from eigenvalues of r_Z

Examples:

* $\mathbb{C}^2 \otimes \mathbb{C}^2$: $r_Z = Z \otimes I + I \otimes Z$

  eigenvalues $= \{2, 0, 0, -2\} = \{2, 0, -2\} \cup \{0\}$ ✓

* $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$: eigenvalues of $r_Z = \{3, 1, 1, 1, -1, -1, -1, -3\}$

  $= \{3, 1, -1, -3\} \cup \{1, -1\} \cup \{1, -1\}$

* Clebsch-Gordan rule: For $k > 0$,

$$\boxed{\mathrm{Sym}^k(\mathbb{C}^2) \otimes \mathbb{C}^2 \cong \mathrm{Sym}^{k+1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{k-1}(\mathbb{C}^2)}$$

$j \otimes \frac{1}{2} = (j+\frac{1}{2}) \oplus (j-\frac{1}{2})$

WILL USE NEXT MONTH!

Pf: $r_Z = r_Z^{(k)} \otimes I + I \otimes Z$

eigenvalues $= \{k, k-2, \ldots, -k\} + \{\pm 1\}$

$= \{k+1, k-1, \ldots, -k-1\} \cup \{k-1, k-3, \ldots, -k+1\}$ $\square$

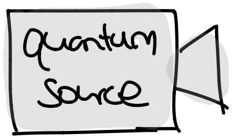
Sanity-check: $(\mathbb{C}^2)^{\otimes 3} = (\mathrm{Sym}^2(\mathbb{C}^2) \oplus \mathrm{Sym}^0(\mathbb{C}^2)) \otimes \mathbb{C}^2$

$= \mathrm{Sym}^3(\mathbb{C}^2) \oplus \mathrm{Sym}^1(\mathbb{C}^2) \oplus \mathrm{Sym}^1(\mathbb{C}^2)$ AS ABOVE

Density Operators

So far: $|\psi\rangle \in \mathcal{H}$, $\psi := |\psi\rangle\langle\psi|$ "pure state"

How to describe an <u>ensemble</u> $\{p_i, |\psi_i\rangle\}$?



emitting $|\psi_i\rangle$ with probability p_i

If $\{Q_x\}$ POVM: $\Pr(\text{outcome } x) = \sum_i p_i \langle\psi_i|Q_x|\psi_i\rangle$

$$= \sum_i p_i \operatorname{tr}\left[|\psi_i\rangle\langle\psi_i| Q_x\right] = \operatorname{tr}\Big[\underbrace{\sum_i p_i |\psi_i\rangle\langle\psi_i|}_{=:\,\rho} \cdot Q_x\Big]$$

- $\rho \geq 0$
- $\operatorname{tr}\rho = 1$

Any such object is called density operator. "density matrix"

* <u>Born's rule</u>: $\Pr_\rho(\text{outcome } x) = [\operatorname{tr} \rho Q_x]$

* If $X = \sum_x x P_x$ observable: $E_\rho[\text{outcome}] = \operatorname{tr}[\rho X]$

Outcome $x \longrightarrow$ post-meas. state $\rho' = \dfrac{P_x \rho P_x}{\operatorname{tr}[P_x \rho]}$ $\boxed{\text{PSET}}$

$\ulcorner$ If $\rho = |\psi\rangle\langle\psi|$: $\rho' = |\psi'\rangle\langle\psi'|$, where $|\psi'\rangle = \dfrac{P_x|\psi\rangle}{\|P_x|\psi\rangle\|}$ ✓ $\lrcorner$

* Any ρ arises from ensemble (spectral decomposition)

↳ $\boxed{\text{density operators} = \text{(general) quantum states}}$

* <u>WARNING</u>: Ensemble is <u>not</u> unique!

$$\frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \overset{!}{=} \frac{1}{2}\left(|+\rangle\langle +| + |-\rangle\langle -|\right) = \frac{I}{2}$$

$$= \frac{1}{4}\left(|0\rangle\langle 0| + |1\rangle\langle 1| + |+\rangle\langle +| + |-\rangle\langle -|\right)$$

Probabilities are <u>not</u> necessarily eigenvalues of ρ
(only if states in ensemble are orthogonal)

* <u>Pure state</u>:

$$\rho = |\psi\rangle\langle\psi| \iff \mathrm{rk}(\rho) = 1 \iff \text{eigenvalues } \{1, 0, \dots, 0\}$$

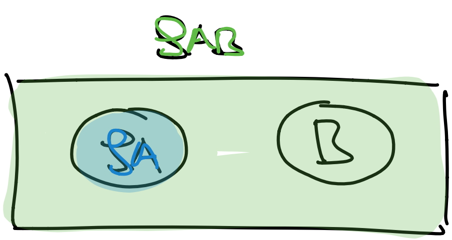
$$\iff \rho^2 = \rho$$

Otherwise: <u>mixed state</u>

<u>Applications</u>: information sources, statistical physics (Gibbs states...), classical probability:

$$\{p_i, |i\rangle\} \rightarrow \sum_i p_i \, |i\rangle\langle i|$$

... but also <u>subsystems</u>!



Consider ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$, $\{Q_{A,x}\}$ POVM on A:

$$\Pr_\rho(\text{outcome } x) = \mathrm{tr}\left[\rho_{AB}\left(Q_{A,x} \otimes I_B\right)\right]$$

$$= \sum_{a,b} \underbrace{\langle ab| \varrho_{AB} (Q_{A,x} \otimes I_B) |ab\rangle}_{\langle a| (I_A \otimes \langle b|) \varrho_{AB} (I_A \otimes |b\rangle) Q_{A,x} |a\rangle}$$

$$= \operatorname{tr}\Big[\underbrace{\boxed{\sum_b (I_A \otimes \langle b|) \varrho_{AB} (I_A \otimes |b\rangle)}}_{=: \operatorname{tr}_B[\varrho_{AB}]} Q_{A,x} \Big]$$

partial trace

makes sense for arbitrary operators M_{AB}

* We call $\varrho_A := \operatorname{tr}_B[\varrho_{AB}]$ the reduced density operator of ϱ_{AB} on subsystem A   Indeed a density operator.

$$\Longrightarrow \boxed{\Pr_\varrho(\text{outcome } x) = \operatorname{tr}[\varrho_A Q_{A,x}]}$$

describes state of subsystem

Rules: * $\operatorname{tr}[M_{AB}(X_A \otimes I_B)] = \operatorname{tr}[\operatorname{tr}_B[M_{AB}] \cdot X_A]$

* $\operatorname{tr}_B[M_A \otimes N_B] = M_A \cdot \operatorname{tr}[N_B]$ "partial trace"

Example: $|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$\Rightarrow \rho_{AB} = \frac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right)$$

$$= \frac{1}{2}\Big(\underbrace{|00\rangle\langle 00|}_{=|0\rangle\langle 0| \otimes |0\rangle\langle 0|} + \underbrace{|00\rangle\langle 11|}_{=|0\rangle\langle 1| \otimes |0\rangle\langle 1|} + |11\rangle\langle 00| + |11\rangle\langle 11|\Big)$$

COMPARE LECTURE 2

$$\Rightarrow \rho_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

* WARNING: Even if ρ_{AB} pure, ρ_A can be mixed!