Today's goal: Estimating a pure state

$$|\psi\rangle \xrightarrow[???]{\text{meas.}} \text{"}\psi\text{"}$$

q. state — classical description

- $|\psi\rangle \longrightarrow \text{"}\psi\text{"}$ ↯ could distinguish non-orthogonal states

- $\left(\langle\psi|^{\otimes n}\right)\left(|\phi\rangle^{\otimes n}\right) = \langle\psi|\phi\rangle^n \longrightarrow 0$ if $\psi \neq \phi$

  many copies are almost orthogonal !

Goal: Design POVM $\{Q_{\hat\psi}\}$ s.th.

$$|\psi\rangle^{\otimes n} \longrightarrow \boxed{\text{POVM}} \Longrightarrow \hat\psi \approx \psi$$

$\hat\psi$ ↳ estimate

on average / with high probability

Comments:

* We will use the fidelity $|\langle\hat\psi|\psi\rangle|$ to quantify goodness of estimation scheme. → $\boxed{\text{PSET 1}}$

* $|\psi\rangle$ and $e^{i\alpha}|\psi\rangle$ correspond to same state

  NOTATION!!!

  ↳ use $\hat\psi := |\hat\psi\rangle\langle\hat\psi|$ to label outcomes etc.

  "pure quantum state"

will next week learn about more general states

* $\Omega = \{\hat{\psi} = |\psi\rangle\langle\psi|\}$ is a continuum. POVM!?

Continuous POVM: Ω space of outcomes, dx measure

$$\boxed{\begin{array}{l} \bullet\ Q_x \geq 0 \\ \bullet\ \int dx\, Q_x = I \end{array}}$$

need both to specify POVM

Born's rule:

$\langle\psi|Q_x|\psi\rangle$ = prob. density of outcome

i.e.

$$\Pr_\psi(\text{outcome} \in S) = \int_S dx\, \langle\psi|Q_x|\psi\rangle$$

$$E_\psi[f(x)] = \int dx\, \langle\psi|Q_x|\psi\rangle\, f(x)$$

[Note: $\Pr(\Omega) = \int dx\, \langle\psi|Q_x|\psi\rangle = \langle\psi| \underbrace{\int dx\, Q_x}_{=I} |\psi\rangle = 1$]

* Any continuous POVM is physical.

Can be implemented by randomly chosen finite POVM.

* Examples of measures:

unique up to normalization

Ω finite $\rightsquigarrow$ dx = uniform measure — permutation symmetry

$\Omega = \mathbb{R}$ $\rightsquigarrow$ dx = Lebesgue measure — translation symmetry

$\boxed{\Omega = S^n}$ $\rightsquigarrow$ dx = rotationally-inv. measure — rotation symmetry

What measure to use on the set of pure states?

Fact: There exists a unique probability measure $d\psi$ on the set of pure states $\{\psi = |\psi\rangle\langle\psi|\}$ on $\mathbb{C}^d$ s.th.

$$\int d\psi \, f(\psi) = \int d\psi \, f(U\psi U^\dagger)$$

for all unitaries U & all (integrable) fns f.

"Uniform prob distribution", "Haar measure"

How do we come up with a good POVM?

! suggests: $\{\frac{1}{2} |\hat{\psi}\rangle\langle\hat{\psi}|^{\otimes n}\}$ is a good POVM !!!

almost orthogonal for large n

Let's study the symmetries:

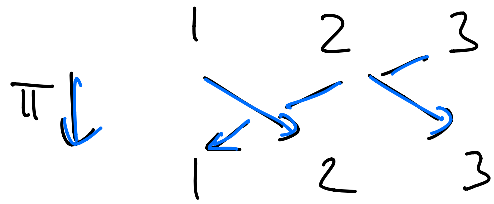
$$|\psi\rangle \in \mathbb{C}^d \longrightarrow \boxed{|\psi\rangle^{\otimes n}} \in (\mathbb{C}^d)^{\otimes n}$$

invariant under permuting subsystems

Symmetric group ??? S_n:

- elements are permutations $\pi : \{1,\dots,n\} \to \{1,\dots,n\}$
- $\# = n!$

For $\pi \in S_n$, define operator on $(\mathbb{C}^d)^{\otimes n}$:



$$R_\pi |\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle = |\psi_{\pi^{-1}(1)}\rangle \otimes \dots \otimes |\psi_{\pi^{-1}(n)}\rangle$$

$$\Longrightarrow \boxed{R_{id} = I \quad \& \quad R_\pi R_\tau = R_{\pi\tau}}$$

???

representation of S_n on $(\mathbb{C}^d)^{\otimes n}$

Moreover: R_π's are unitary.

Symmetric subspace: Bosonic Fock space w/ n particles

$$\mathrm{Sym}^n(\mathbb{C}^d) = \{ |\Phi\rangle \in (\mathbb{C}^d)^{\otimes n},\ R_\pi |\Phi\rangle = |\Phi\rangle \ (\forall \pi) \}$$

$\cup$

$|\psi\rangle^{\otimes n}$

Ex: $\mathrm{Sym}^2(\mathbb{C}^2)$ spanned by

$|00\rangle$, $|11\rangle$, $\frac{|10\rangle + |01\rangle}{\sqrt{2}}$

not of this form!

Missing: $|10\rangle - |01\rangle / \sqrt{2}$.

Symmetrize:

$$\Pi_n = \frac{1}{n!} \sum_{\pi \in S_n} R_\pi$$

projector onto symmetric subspace ✓

- $|\Phi\rangle$ arbitrary $\Rightarrow \Pi_n|\Phi\rangle \in \mathrm{Sym}^n(\mathbb{C}^d)$
- $|\Phi\rangle \in \mathrm{Sym}^n(\mathbb{C}^d) \Rightarrow \Pi_n|\Phi\rangle = |\Phi\rangle$

e.g. $R_\pi(\Pi_n|\Phi\rangle) = \frac{1}{n!}\sum_{\tau\in S_n} R_{\pi\tau}|\Phi\rangle = \frac{1}{n}\sum_{\tilde{\pi}\in S_n} R_{\tilde{\pi}}|\Phi\rangle$
$= \Pi_n|\Phi\rangle$. ✓

- $\Pi_n^\dagger = \Pi_n$

<u>Basis of $\mathrm{Sym}^n(\mathbb{C}^d)$</u>:

$$\Pi_n|\underbrace{1\cdots1}_{t_1 \text{ many}},\underbrace{2\cdots2}_{},\underbrace{3\cdots3}_{},\underbrace{d\cdots d}_{t_d \text{ many}}\rangle \qquad \sum_{i=1}^{d} t_i = n$$

$t = (t_1,\ldots,t_d)$ occupation numbers / type

$$\Rightarrow \dim \mathrm{Sym}^n(\mathbb{C}^d) = \binom{n+d-1}{n} = \frac{(n+d-1)!}{n!\,(d-1)!}$$
$$= \mathrm{tr}\,\Pi_n$$

Recall: • $\langle\psi|^{\otimes n}|\phi\rangle^{\otimes n} \neq 0$ in general

yet: • not all $|\Phi\rangle \in \mathrm{Sym}^n$ of form $|\psi\rangle^{\otimes n}$

Claim:

$$\boxed{\Pi_n = \binom{n+d-1}{n} \underbrace{\int d\psi}_{\text{pue States}} |\psi\rangle^{\otimes n} \langle\psi|^{\otimes n}}$$

PROOF NEXT WEEK

- $|\psi\rangle^{\otimes n}$ is "overcomplete basis": $|\Phi\rangle \in \mathrm{Sym}^n$

$$\Longrightarrow |\Phi\rangle = \Pi_n |\Phi\rangle = (\cdot\cdot) \underbrace{\int d\psi\, |\psi\rangle^{\otimes n} \cdot \underbrace{(\langle\psi|^{\otimes n})|\Phi\rangle}_{\text{Scalar}}}_{\text{linear combination}}$$

- $Q_{\hat\psi} = \binom{n+d-1}{n} |\hat\psi\rangle^{\otimes n} \langle\hat\psi|^{\otimes n}$ is a POVM on Sym^n

$$\int d\hat\psi\, Q_{\hat\psi} = \Pi_n$$

How to quantify performance of this estimator?

$$\boxed{|\langle\hat\psi|\psi\rangle|^{2k}} \quad \text{overlap}$$

over outcome!!!

Average performance:

$$E\, |\langle\hat\psi|\psi\rangle|^{2k}$$

$$\overset{\text{Born}}{=} \int d\hat\psi\, \underbrace{\langle\psi|^{\otimes n} Q_{\hat\psi} |\psi\rangle^{\otimes n}}_{\text{prob density}} |\langle\hat\psi|\psi\rangle|^{2k}$$

$$= \binom{n+d-1}{n} \int d\hat{\psi} \; |\langle\hat{\psi}|\psi\rangle|^{2(n+k)}$$

$$= \binom{n+d-1}{n} \langle\psi|^{\otimes(n+k)} \left[\int d\hat{\psi} \; |\hat{\psi}\rangle^{\otimes(n+k)}\langle\hat{\psi}|^{\otimes(n+k)}\right] |\psi\rangle^{\otimes(n+k)}$$

$$= \binom{n+d-1}{n}\binom{n+k+d-1}{n+k}^{-1} \underbrace{\langle\psi|^{\otimes(n+k)} \Pi_{n+k} |\psi\rangle^{\otimes(n+k)}}_{=1}$$

$$= \frac{(n+d-1)!}{n!\,\cancel{(d-1)!}} \; \frac{(n+k)!\,\cancel{(d-1)!}}{(n+k+d-1)!}$$

$a \leq b \Rightarrow \frac{a+i}{b+i} \geq \frac{a}{b}$

$$= \frac{(n+d-1) \cdots\cdots (n+1)}{(n+k+d-1) \cdots (n+k+1)} \geq \left[\frac{n+1}{n+k+1}\right]^{d-1}$$

$$= \left[1 - \frac{k}{n+k+1}\right]^{d-1} \geq 1 - \frac{(d-1)k}{n+k+1}$$

$$\geq 1 - \frac{dk}{n}.$$

Intuitive! $|\psi\rangle$ has $O(d)$ components

$\Downarrow$

need $n = \Omega(d)$

$k=1$

SUCCESS ! High fidelity if $n \gg d$.

In terms of trace distance:

$$E[T(\psi,\hat{\psi})] = E\left[\sqrt{1-|\langle\hat{\psi}|\psi\rangle|^2}\right]$$

$$\leq \sqrt{E[1-|\langle\hat{\psi}|\psi\rangle|]} \leq \boxed{\sqrt{\frac{d}{n}}}$$

Jensen ineq.

And n meas. of any component should give std. deviation $O(\frac{1}{\sqrt{n}})$