Last time we discussed measurements in QM. Recall:

Observable $O = \sum_{x \in \Omega} x \cdot P_x \longleftrightarrow \{P_x\}_{x \in \Omega}$ projective measurement

no need to restrict to $\Omega \subseteq \mathbb{R}$ (just relabel)

Born rule: $\Pr_\psi(\text{outcome } x) = \langle \psi | P_x | \psi \rangle = \| P_x |\psi\rangle \|^2$

$P_x^2 = P_x$ USEFUL

Measuring subsystems:

$$O_A \text{ on } \mathcal{H}_A \longrightarrow O_A \otimes I_B \text{ on } \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\{P_{A,x}\} \qquad\qquad \{P_{A,x} \otimes I_B\}$$

NOTATION!

Ex: Max. entangled state ("ebit"):

$$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right)$$

e.g. $|0\rangle, |1\rangle$ or $|+\rangle, |-\rangle$

Let $P_{A,x} = |e_x\rangle\langle e_x|_A$ arbitrary basis of $\mathbb{C}^2$

$$\Pr(\text{outcome } x) = \langle \Phi^+_{AB} | \left(P_{A,x} \otimes I_B \right) | \Phi^+_{AB} \rangle$$

$$= \frac{1}{2} \sum_{i,j} \left(\langle i|_A \otimes \langle i|_B \right) \left(P_{A,x} \otimes I_B \right) \left(|j\rangle_A \otimes |j\rangle_B \right)$$

$$= \frac{1}{2} \sum_i \langle i | P_{A,x} | i \rangle = \frac{1}{2} \operatorname{tr}[P_{A,x}] = \frac{1}{2}$$

$\boxed{\text{50\% / 50\% FOR ANY MEAS.}}$

Entanglement as a resource

Two communication scenarios where entanglement helps.
Clarify bits vs. qubits.

(1) Encoding bits into qubits

How many bits can we send by sending a qubit?

Sender: Alice

encode message $m \in \{0, \ldots, M-1\}$ into qubit state $\boxed{|\psi_m\rangle \in \mathbb{C}^2}$

Receiver: Bob

receives $|\psi_m\rangle \in \mathbb{C}^2$, performs measurement to deduce m.

* Only possibly perfectly if $M \leq 2$ — need $\langle \psi_x | \psi_y \rangle = \delta_{xy}$

* $M = 2$ (a single bit) is possible: $|\psi_m\rangle := |m\rangle$

Superdense coding: Can do better using entanglement!

Consider following orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$|\phi_0\rangle = \sqrt{\tfrac{1}{2}}\left(|00\rangle + |11\rangle\right) = (I \otimes I)\,|\Phi^+\rangle$$

$$|\phi_1\rangle = \sqrt{\tfrac{1}{2}}\left(|00\rangle - |11\rangle\right) = (Z \otimes I)\,|\Phi^+\rangle$$

$$|\phi_2\rangle = \sqrt{\tfrac{1}{2}}\left(|10\rangle + |01\rangle\right) = (X \otimes I)\,|\Phi^+\rangle$$

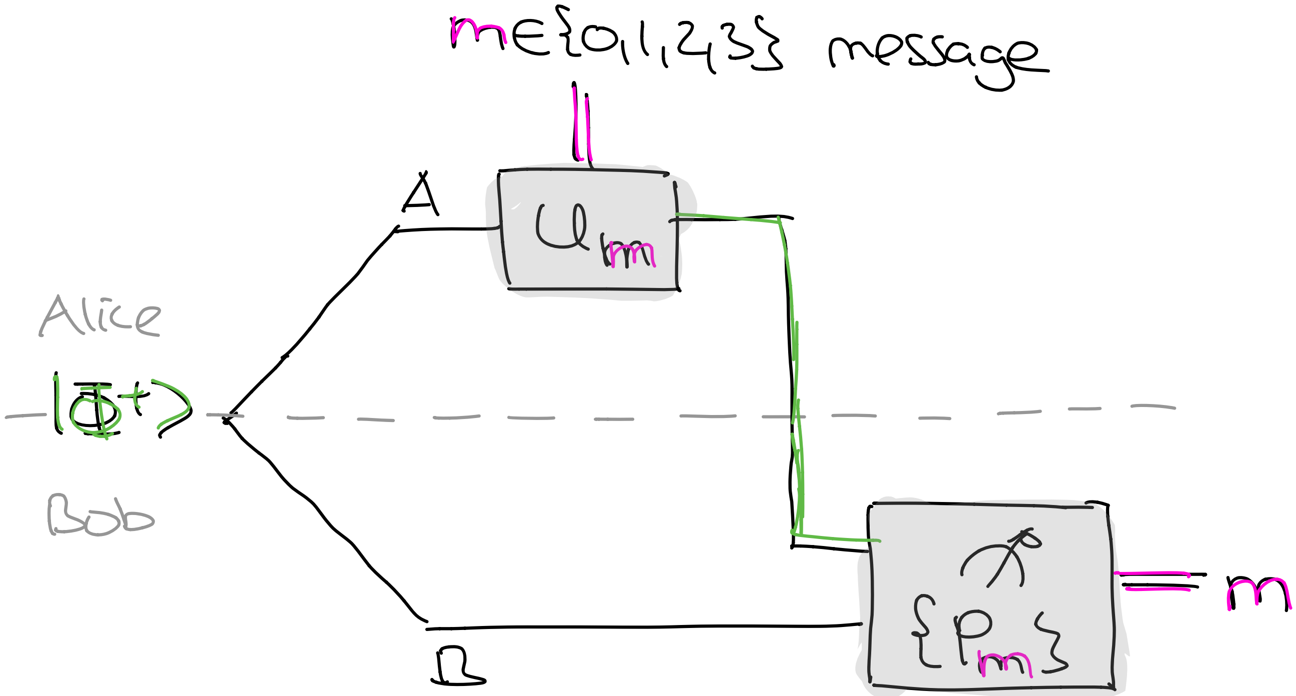
$$|\phi_3\rangle = \sqrt{\tfrac{1}{2}}\left(|10\rangle - |01\rangle\right) = (XZ \otimes I)\,|\Phi^+\rangle$$

i.e., $|\phi_m\rangle_{AB} = (U_{A,m} \otimes I_B)\,|\Phi^+_{AB}\rangle$

can create locally from ebit!!!

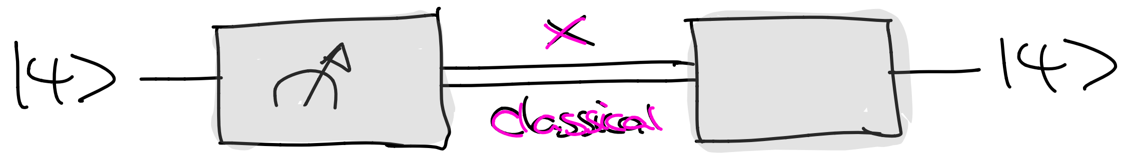
Protocol:

- Alice & Bob share ebit $|\Phi^+\rangle_{AB}$
- Alice applies $U_{A,m}$ to her qubit & sends it over.
  her message
  one qubit, but now Bob has two
- Bob measures $\{P_{AB,m} := |\phi_m\rangle\langle\phi_m|_{AB}\}$



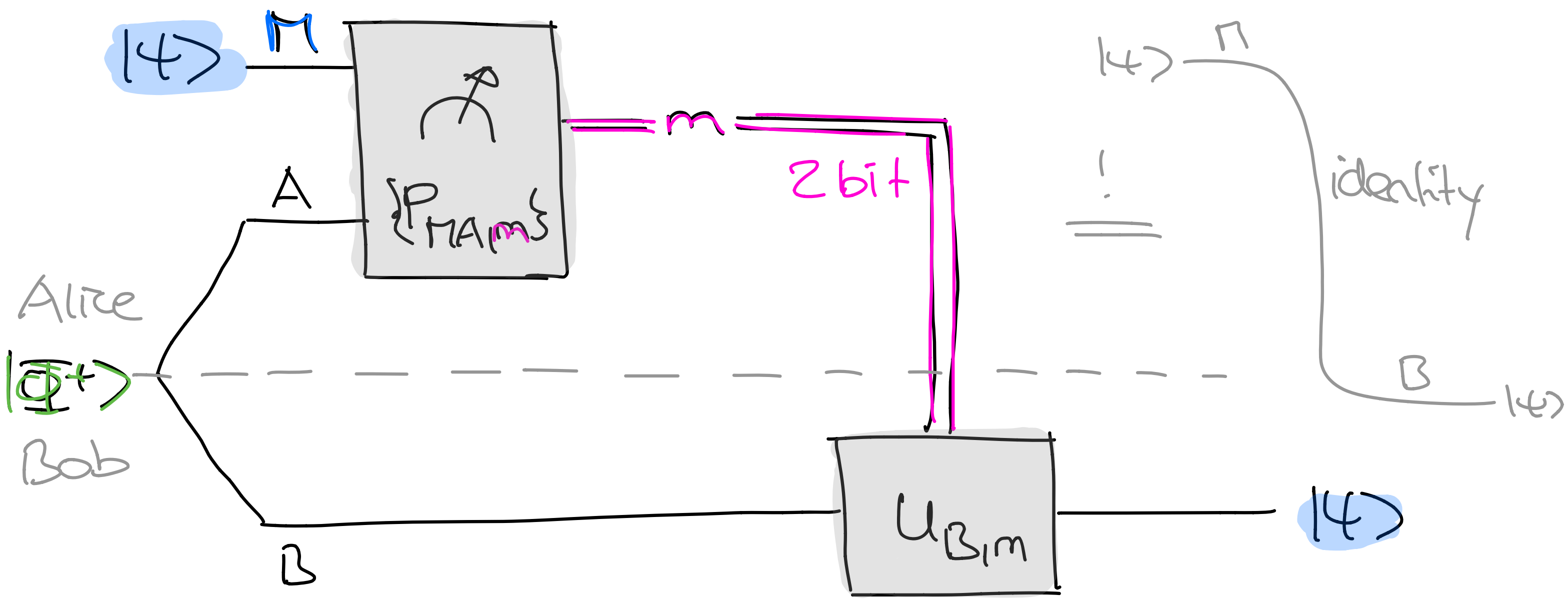
Can we also go the other way around!

② Encoding qubits into bits



IMPOSSIBLE: Measurement ~~cannot~~ perfectly distinguish non-orthogonal $|\psi\rangle$. Also, infinitely many $|\psi\rangle$, but only finitely many bit-strings!

Teleportation: Can do it using entanglement



- Initial state: $|\psi\rangle_M \otimes |\Phi^+\rangle_{AB}$

- Alice measures $P_{MA,m} = |\phi_m\rangle\langle\phi_m|_{MA}$ where

$$|\phi_m\rangle_{MA} = (U_{M,m} \otimes I_A)\, |\Phi^+_{MA}\rangle \qquad \text{SEE ABOVE}$$

Probability of outcomes? Post-meas. state?

Useful calculation:

$$\left(\langle\phi_m|_{MA}\otimes I_B\right)\left(|\psi\rangle_M\otimes|\Phi^+_{AB}\rangle\right)$$

$$=\left(\langle\Phi^+|_{MA}\otimes I_B\right)\left(U^\dagger_{M,m}|\psi_M\rangle\otimes|\Phi^+_{AB}\rangle\right)$$

$$=\boxed{\left(\langle\Phi^+_{MA}|\otimes I_B\right)\left(I_M\otimes|\Phi^+_{AB}\rangle\right)}\,U^\dagger_{M,m}|\psi_M\rangle$$

$$=\frac{1}{2}\sum_{x,y}\left(\langle x|_M\otimes\langle x|_A\otimes I_B\right)\left(I_M\otimes|y\rangle_A\otimes|y\rangle_B\right)$$

$$=\frac{1}{2}\boxed{\sum_x |x\rangle_B\langle x|_M}$$

IDENTITY FROM M TO B

$$=\frac{1}{2}U^\dagger_{B,m}|\psi_B\rangle$$

!!!

- **Post-meas. state:** Proportional to

$$\left(P_{MA,m}\otimes I_B\right)\left(|\psi\rangle_M\otimes|\Phi^+_{AB}\rangle\right)$$

$$=\frac{1}{2}|\phi_m\rangle_{MA}\otimes U^\dagger_{B,m}|\psi_B\rangle$$

- **Bob** applies $U_{B,m}$ $\rightsquigarrow$ $\frac{1}{2}|\phi_m\rangle_{MA}\otimes|\psi_B\rangle$

SUCCESS !

* **Note:** $\Pr(\text{outcome } m) = \|\cdots\|^2 = \frac{1}{4}$

Alice learns nothing !

* If we apply protocol to part of $|\psi_{ME}\rangle$
  ↳ obtain $|\psi_{BE}\rangle$ ENTANGLEMENT SWAPPING $\boxed{\text{PSET}}$

Resource inequalities

- $\underbrace{[q\to q]}_{\text{sending 1 bit}} \overset{\text{local ops}}{\geq} \underbrace{[c\to c]}_{\text{sending 1 qubit}}$ Alice encodes m by $|m\rangle$, Bob measures $\{|m\rangle\langle m|\}$
- $[q\to q] \not\geq 2\,[c\to c]$ see above
- Super-dense coding: $\text{ebit} + [q\to q] \geq 2\,[c\to c]$

- $n\,[c\to c] \not\geq [q\to q]$ see above
- Teleportation: $\text{ebit} + 2[c\to c] \geq [q\to q]$

Together: $\boxed{[q\to q] \equiv 2[c\to c] \text{ mod ebit}}$

- $[q\to q] \geq \text{ebit}$ prepare locally & send qubit over
- $n\ \text{ebit} \not\geq \begin{matrix}[q\to q]\\ [c\to c]\end{matrix}$ CANNOT COMMUNICATE USING ENTANGLEMENT ALONE

Generalized Measurements

Observable $O = \sum_{x\in\Omega} x \cdot P_x \quad \longleftrightarrow \quad \{P_x\}_{x\in\Omega}$ projective measurement

More general measurements are possible!

ancillary system

notation!!!

$$\Pr_\psi(\text{outcome } x) = \left(\langle\psi|_A \otimes \langle 0|_B\right) P_{AB,x} \left(|\psi\rangle_A \otimes |0\rangle_B\right)$$

$$= \langle\psi_A| \underbrace{\left[\left(I_A \otimes \langle 0|_B\right) P_{AB,x} \left(I_A \otimes |0\rangle_B\right)\right]}_{=:\, Q_{A,x}} |\psi_A\rangle$$

$$= \langle\psi_A| Q_{A,x} |\psi_A\rangle$$

Note: $Q_{A,x} \geq 0$ & $\sum_{x\in\Omega} Q_{A,x} = I_A$

$$= \left(I_A \otimes \langle 0|_B\right) \overbrace{\sum_x P_{AB,x}}^{= I_{AB}} \left(I_A \otimes |0\rangle_B\right) \checkmark$$

$\boxed{\text{Def: A collection of operators } \{Q_x\} \text{ on } \mathcal{H} \text{ is called a POVM if } Q_x \geq 0 \ (\forall x) \text{ and } \sum_x Q_x = I.}$

* Born's rule: $\Pr_\psi(\text{outcome } x) = \langle\psi|Q_x|\psi\rangle$

* Truely more general: E.g.,

$$\{\underbrace{\tfrac{1}{2}|0\rangle\langle 0|}_{\text{not a projection}}, \tfrac{1}{2}|1\rangle\langle 1|, \tfrac{1}{2}|+\rangle\langle +|, \tfrac{1}{2}|-\rangle\langle -|\}$$

not orthogonal

50% → Z-meas.

→ X-meas.

→ $\boxed{\text{PSET}}$

* Every POVM is physical (i.e. can be realized as above).

BUT: Only specifies probabilities, not post-meas. state.

for this need to know the implementation

* Most general "memoryless" measurement allowed by QM.

w/ finitely many outcomes!

STILL TRUE: Can only distinguish orthogonal states perfectly!!! POVMs do not help

$\boxed{\text{PSET}}$