Last time: Quantum Schur transform

key ingredient, still need to construct!

## A circuit for the Clebsch-Gordan transform

$$J_k : \mathrm{Sym}^k(\mathbb{C}^2) \otimes \mathbb{C}^2 \cong \bigoplus_{p=\pm 1} \mathrm{Sym}^{k+p}(\mathbb{C}^2)$$

CG coeffs

How to construct? Recall how we proved $\cong$ ...

Key tool: $\{R_u\} \rightsquigarrow r_z := -i\partial_{s=0}\left[R_{e^{isz}}\right]$

* Why? $J_k r_z^{LHS} J_k^\dagger = r_z^{RHS} \longrightarrow$ preserves eigenspaces

So let's diagonalize $r_z^{LHS}$ & $r_z^{RHS}$ ...

* $\mathrm{Sym}^k(\mathbb{C}^2)$: $T_u^{(k)} \rightsquigarrow t_z^{(k)} =$ restr. of $z\otimes I \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I \otimes z$

NOTATION: $|k,s\rangle := |\omega_{m,k-m}\rangle$ eigenvectors

$s := 2m-k \in \{k, k-2, \ldots, -k\}$ eigenvalues

* $\mathrm{Sym}^k(\mathbb{C}^2) \otimes \boxed{\mathbb{C}^2}$: $t_z^{(k)} \otimes I + I \otimes t_z^{(1)} =: r_z^{LHS}$ (discuss)

$|k,s\rangle \otimes |b\rangle \rightsquigarrow$ eigenvalue $s \pm (-1)^b$

* $\mathrm{Sym}^{k+1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{k-1}(\mathbb{C}^2)$: $t_z^{(k+1)} \oplus t_z^{(k-1)} =: r_z^{RHS}$

$|k+1, s'\rangle$, $|k-1, s'\rangle \rightsquigarrow$ eigenvalue $s'$

$$J\left(|k, s' - (-1)^b\rangle \otimes |b\rangle\right) = \sum_{p=\pm 1} U(k,s)_{p,b}\, |k+p, s'\rangle$$

2x2 unitary matrix :)

* rewrite:

(also possible for $s' = \pm(k+1)$)

$$J\left(|k,s\rangle \otimes |b\rangle\right) = \sum_{p=\pm 1} U(k, s+(-1)^b)_{p,b}\, |k+p, s+(-1)^b\rangle$$

* How to compute?

$$J\left(|k,k\rangle \otimes |0\rangle\right) = |k+1, k+1\rangle \overset{r_{\Pi_-}}{\rightsquigarrow} |k+1, k-1\rangle$$

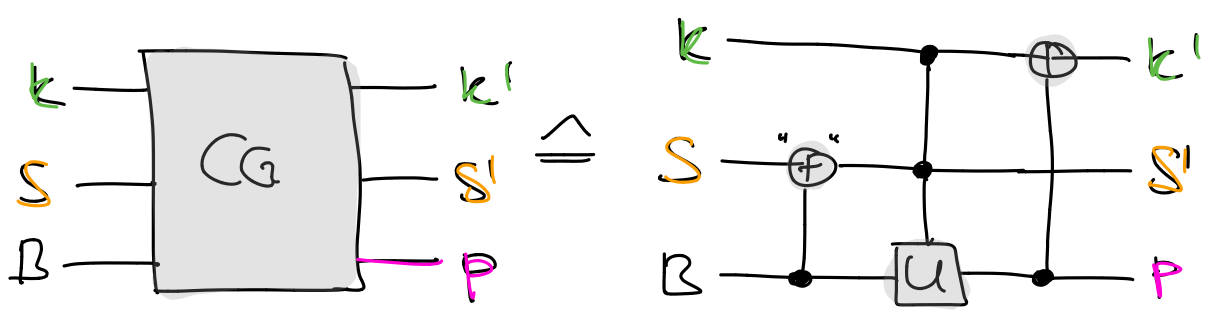
$$\overset{\perp}{\rightsquigarrow} |k-1, k-1\rangle$$

& continue via $r_{\Pi_-}$

<u>Result</u>:

$\langle k', s' | J_k\left(|k,s\rangle \otimes |b\rangle\right)$ ?

$$|k\rangle \otimes |s\rangle \otimes |b\rangle \longrightarrow \sum_p |k+p\rangle \otimes |s+(-1)^b\rangle \otimes \overbrace{U(k, s+(-1)^b)_{p,b}\, |p\rangle}$$




<u>Rest of today</u>: Q-entropy & a new task...

# Entropy & Entropy inequalities

**Shannon entropy:** $H(\{p_i\}_{i=1}^{d}) := -\sum_{i=1}^{d} p_i \cdot \log p_i$

$0 \cdot \log 0 := 1$

* $H(\{p, 1-p\}) = h(p)$ ✓

**Von Neumann entropy:** $\rho$ on $\mathbb{C}^d$

$$S(\rho) := -\mathrm{tr}[\rho \cdot \log \rho]$$

* $S(\rho) = H(\{p_i\})$ if $\{p_i\}$ eigenvalues (with multiplicity!)
* $0 \leq S(\rho) \leq \log d$, $= \log d$ iff $\rho = \frac{I}{d}$

  Jensen: $\sum_i p_i \log\left(\frac{1}{p_i}\right) \leq \log\left(\sum_i p_i \frac{1}{p_i}\right)$

* Optimal asympt. rate for compression, state transfers also if $d \geq 2$ (we only did qubits)

* Asymptotic equipartition property (AEP):
  $\forall \varepsilon > 0$ $\exists$ "typical" projectors $\{P_n\}$ on $(\mathbb{C}^d)^{\otimes n}$

  ① $\mathrm{tr}[P_n \rho^{\otimes n}] \longrightarrow 1$

  ② $\mathrm{rk}[P_n] \leq 2^{n(S(\rho)+\varepsilon)}$

  ③ eigenvalues of $P_n \rho P_n$ are $2^{-n(S(\rho) \pm \varepsilon)}$

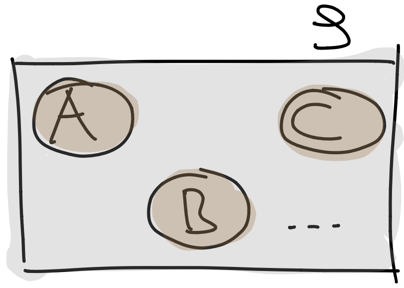
  We / you proved this for $d=2$! (③? PSET 6 :-)

**Summary:** $\rho^{\otimes n} \overset{\text{gentle meas.}}{\approx} P_n \rho^{\otimes n} P_n \leftarrow$ "uniform" on typ. subspace

# Entropies of subsystems:

* NOTATION: $S(A)_\rho := S(\rho_A)$ for $\rho_{ABC\ldots}$

  ↑ often omit




* $\rho_{AB}$ pure: $S(AB)=0$, $\boxed{S(A)=S(B)} = S_E$

  ↕ compare

* $\rho_{AB}=\rho_A\otimes\rho_B$: $S(AB)=S(A)+S(B)$

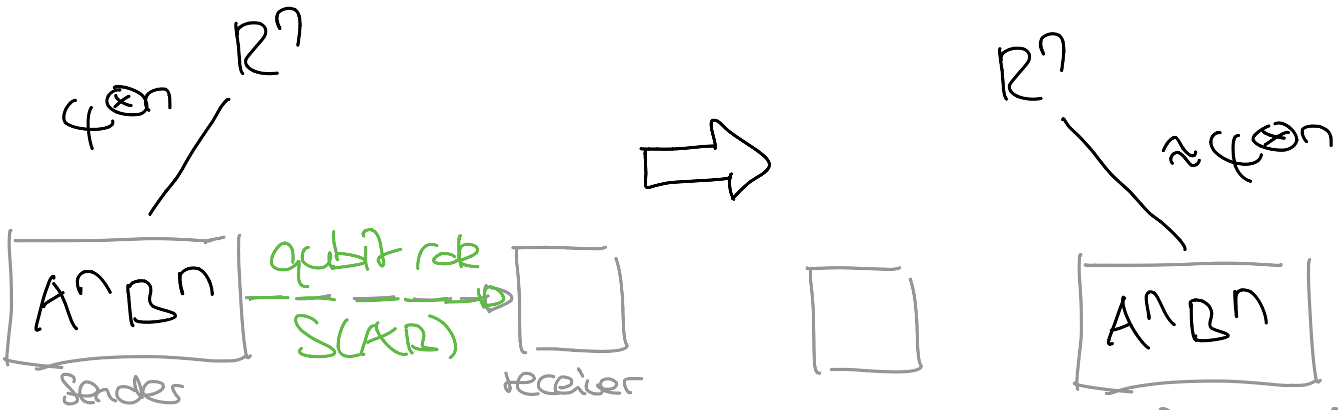
  Pf: $\rho_A\{p_i\}$, $\rho_B\{q_j\} \longrightarrow \rho_{AB}\{p_iq_j\}$ &

  $$-\sum_{i,j} p_iq_j\log(p_iq_j) = -\sum_{i,j}p_iq_j\log(p_i) - \sum_{i,j}p_iq_j\log(q_j) \quad \square$$

* Subadditivity: $S(AB)_\rho \le S(A)_\rho + S(B)_\rho$

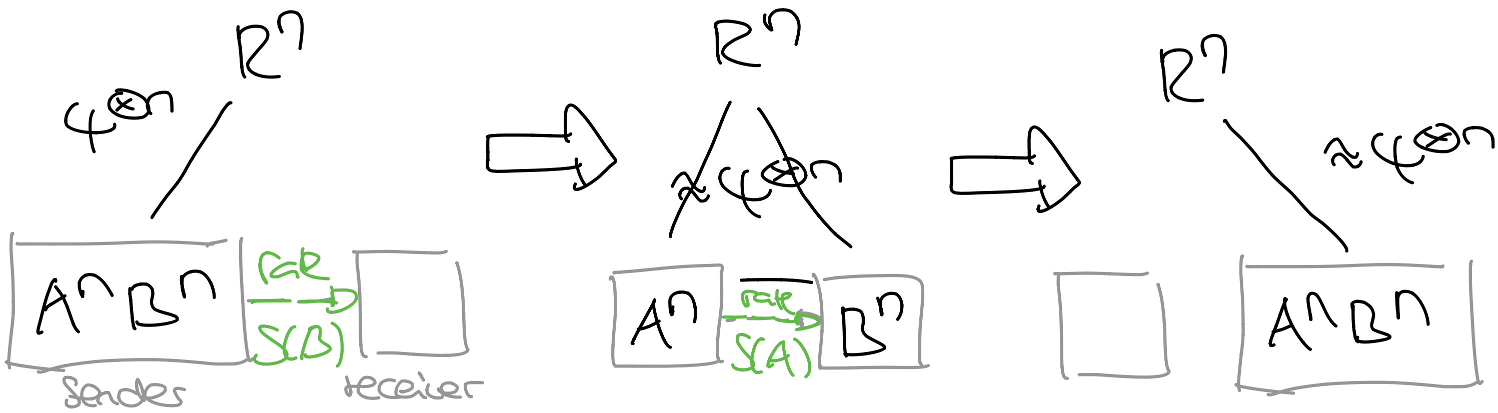
  "Pf": Let $|\psi_{ABR}\rangle$ be a purification.

  Recall q. state transfer.




• $S(AB)$ is optimal qubit rate ← we did not fully prove this (otherwise completely rigorous)

• ... but $S(A)+S(B)$ is <u>achievable</u>!




treat $A^n$ as if not available — treat $B^n$ as not available

Alternative analysis proof: Klein's inequality

($\square$)

* Araki–Lieb inequality:

$$S(AB) \geq |S(A) - S(B)|$$

Pf: $\rho_{AB} \rightsquigarrow |\psi_{ABR}\rangle$

$$S(AB) = S(C) \overset{SA}{\geq} S(BC) - S(B) = S(A) - S(B) \text{ etc. } \square$$

## Mutual information: $\rho_{AB}$ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$:

$$I(A:B)_\rho := S(A)_\rho + S(B)_\rho - S(AB)_\rho$$

$\hat{=}$ info that we lose when we treat A+B independently

* $I(A:B) \overset{SSA}{\geq} 0$, $=0$ iff $\rho_{AB} = \rho_A \otimes \rho_B$

* $\rho_{AB}$ pure: $\frac{1}{2} I(A:B) = S(A) = S(B) = S_E$

* $I(A:B) \le 2\cdot S(A) \le 2\cdot \log d_A$    likewise for B

Araki-Lieb

* $I(A:B) > S(A) \Longrightarrow \rho_{AB}$ is entangled !

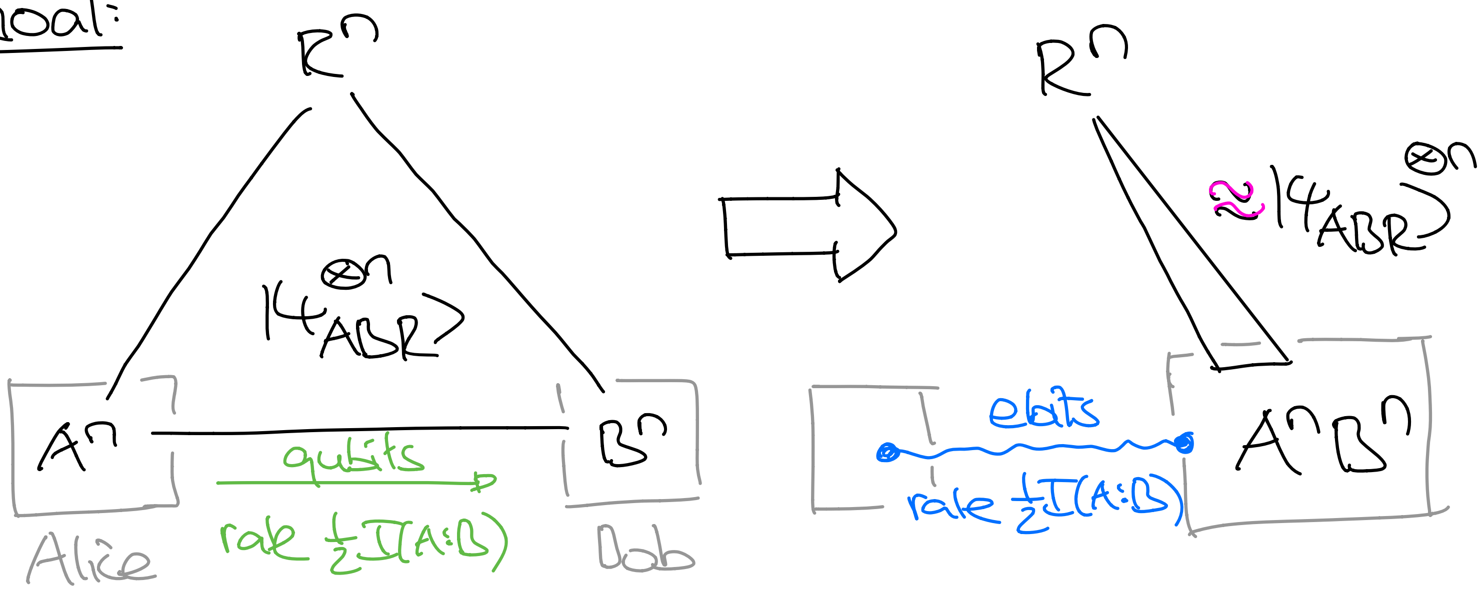
e.g. $|\Phi^+\rangle = \sqrt{\tfrac{1}{2}}\left(|00\rangle + |11\rangle\right)$ : $\quad I(A:B) = 1+1-0 = 2$

$\rho_{AB} = \tfrac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$: $\quad I(A:B) = 1+1-1 = 1$

## Quantum State merging

a.k.a. coherent q. state merging, fully q. Slepian-Wolf, ...

Goal:




* For comparison: q. state transfer needs qubit rate $S(A)$: yields no ebits

* If no B: Q. state transfer $\longrightarrow S(A) = \frac{1}{2} I(A:R)$, optimal!

* If no R: Need not send anything!    Bob can just prepare state locally

↳ Can hope to extract entanglement...

RESULT (tomorrow): ∃ protocol s.th. given $|\psi\rangle_{ABR}$ on

1. qubit rate $\frac{1}{2} I(A:R)$ suffices
2. get ebits at rate $\frac{1}{2} I(A:B)$ for free

generalizes many QIT protocols!