Yesterday: ρ with eigenvalues $\{p, 1-p\}$ $\rightarrow$ How to estimate?

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_k V_{n,k} \otimes \mathbb{C}^{m(n,k)}$$

$$X^{\otimes n} \cong \bigoplus_k \underbrace{T_X^{(n,k)}}_{U(2)\text{-irrep}} \otimes I_{m(n,k)} \quad \leftarrow \forall X$$

* $T_X^{(n,k)} := (\det X)^{(n-k)/2}\, T_X^{(k)}$

* $P_{n,k}$:= projection onto k-th block & $\hat{p} = \frac{1}{2}\left(1 + \frac{k}{n}\right)$

$$\mathrm{tr}[\rho^{\otimes n} P_{n,k}] \leq (n+1) \cdot 2^{-n \underbrace{\boxed{\delta(\hat{p} \| p)}}_{\text{relative entropy}}} \leq (n+1)\, 2^{-n \frac{2}{\ln 2}(\hat{p} - p)^2}$$

RESULT: $\hat{p} \approx p$ with high probability

$$\Pr_{\rho^{\otimes n}}(|\hat{p} - p| \geq \varepsilon) = \sum_{k:\dots} (n+1) 2^{-n\dots} \leq \underbrace{(n+1)^2\, 2^{-n \frac{2}{\ln 2} \varepsilon^2}}_{\text{exponentially small}}$$

Another interpretation: Define

$$P_n := \sum_{k : |\hat{p} - p| < \varepsilon} P_{n,k} \quad \leftarrow \text{projector}$$

* $\mathrm{tr}[P_n \rho^{\otimes n}] = \Pr(|\hat{p} - p| < \varepsilon) \longrightarrow 1$

↳ A typical subspace! What's the rate?

* $\operatorname{rank}(P_n) = \sum_{k : |\hat{p} - p| < \varepsilon} \dim(V_{n,k}) \cdot m(n,k)$

$$\leq \sum_{k : |\hat{p} - p| < \varepsilon} (k+1)\, 2^{n h(\hat{p})} \leq (n+1)^2\, 2^{n(h(p) + \varepsilon')}$$

* P_n only depends on p !

↳ UNIVERSAL typical subspace w/ optimal rate $h(p) = S(\rho)$

↳ protocols for q. data compression & state transfer that work for all ρ w/ spectrum $\{p, 1-p\}$

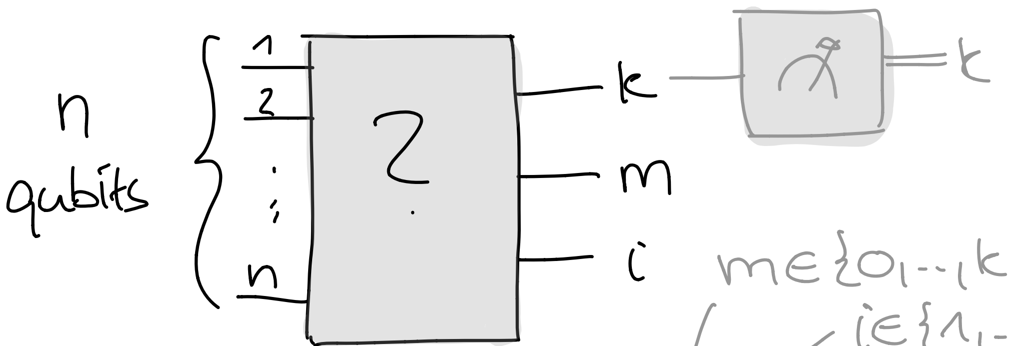
Simple modification: Universal compressor for fixed rate R>0

Rest of today: More about

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_k V_{n,k} \otimes \mathbb{C}^{m(n,k)}$$

* How do we implement this in practice?

Quantum Schur transform = q. algo that implements



implements spectrum estimation

Lecture 13/16 ?

$m \in \{0, \dots, k\}$

$i \in \{1, \dots, m(n,k)\}$

$\{|x_1\rangle \otimes \dots \otimes |x_n\rangle\}$ $\{|k, m, i\rangle\}$ NOT a $\otimes$

How does S_n act? Let's prove generalisation of Schur's lemma:

<u>Lemma</u>: Let $\{V_\lambda\}$ pairwise inequivalent irreps of group G.

① Let $J: V_\lambda \otimes \mathbb{C}^{m(\lambda)} \rightarrow V_\mu \otimes \mathbb{C}^{m(\mu)}$ intertwiner.

If $\lambda \neq \mu$: $J=0$. If $\lambda=\mu$: $J = I_{V_\lambda} \otimes J_\lambda$ arbitrary op.

② Let J be intertwiner <u>on</u> $\bigoplus_\lambda V_\lambda \otimes \mathbb{C}^{m(\lambda)}$.

Then $J = \bigoplus_\lambda I_{V_\lambda} \otimes J_\lambda$ ditto

<u>Proof</u>: ① Consider "components"

$$J_{ab} := (I_{V_\mu} \otimes \langle a|) J (I_{V_\lambda} \otimes |b\rangle) : V_\lambda \rightarrow V_\mu$$

↳ intertwiner between irreps! Schur's lemma:

- $\lambda \neq \mu$: $J_{ab} = 0 \quad \forall a,b \Rightarrow J=0$ ✓
- $\lambda = \mu$: $J_{ab} \propto I_{V_\lambda}$ <u>Define</u> J_λ by $J_{ab} = \langle a|J_\lambda|b\rangle \cdot I_{V_\lambda}$

$$\Rightarrow J = \sum_{a,b} J_{ab} \otimes |a\rangle\langle b| = \sum_{a,b} I_{V_\lambda} \otimes |a\rangle\underbrace{\langle a|J_\lambda|b\rangle}\langle b|$$

$$= I_{V_\lambda} \otimes J_\lambda \checkmark$$

② Apply part ① to "blocks" of

$$J: \bigoplus_\lambda V_\lambda \otimes \mathbb{C}^{m(\lambda)} \rightarrow \bigoplus_\mu V_\mu \otimes \mathbb{C}^{m(\mu)}. \checkmark \quad \square$$

Apply to $G = U(2)$ and $J = R_\pi$ on $(\mathbb{C}^2)^{\otimes n} \cong \bigoplus \ldots$:

Thus: $R_\pi \cong \bigoplus_k I_{V_{n,k}} \otimes R_\pi^{(n,k)}$

* $[R_\pi, P_{n,k}] = 0$ as claimed last time $\Leftarrow$ both symmetries
* The operators $\{R_\pi^{(n,k)}\}_{\pi \in S_n}$ define a representation of S_n on $\mathbb{C}^{m(n,k)} =: W_{n,k}$
* CLAIM: $W_{n,k}$ are pairwise inequivalent irreps — proof later

↳ SCHUR–WEYL DUALITY: As $U(2) \times S_n$-rep.:

pairwise inequiv. $U(2)$-irreps

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_k V_{n,k} \otimes W_{n,k}$$

pairwise ineq. S_n-irreps

$$X^{\otimes n} \cong \bigoplus_k T_X^{(n,k)} \otimes I_{W_{n,k}}$$

$$R_\pi \cong \bigoplus_k I_{V_{n,k}} \otimes R_\pi^{(n,k)}$$

example? $n=3$ on PSET

$$P_{n,k} \cong \bigoplus_{k'} \delta_{kk'} I_{V_{n,k'}} \otimes I_{W_{n,k'}}$$

* If $[M, U^{\otimes n}] = [M, R_\pi] = 0 \ \forall U, \pi$: $\boxed{M = \sum_k z_k \cdot P_{n,k}}$ for suitable $z_i \in \mathbb{C}$

Pf: Lem. for $U(2)$: $M = \bigoplus_k I_{V_{n,k}} \otimes Y_k$ $\Rightarrow$ $X_k \propto I_{V_{n,k}}$ & $Y_k \propto I_{W_{n,k}}$

Lem. for S_n: $M = \bigoplus_k X_k \otimes I_{W_{n,k}}$

$\square$

* $\{P_{n,k}\}$ is finest measurement w/ desired symmetries (up to trivial refinement)
* $V_{n,k} \otimes W_{n,k}$ are irreps of $U(2) \times S_n$

Lemma! ... and any irrep of form $U(2)$-irrep $\otimes$ S_n-irrep

Proof of the claim

* Lemma: If $[R_\pi, M] = 0$ $\forall \pi$: Linear comb. of $X^{\otimes n}$'s

Proof: Will show that $\forall Y$:

did NOT prove this in class

$$\sum_\pi R_\pi Y R_\pi^\dagger \text{ is linear combinat. of } X^{\otimes n}\text{'s}$$

WLOG: $Y = Y_1 \otimes \cdots \otimes Y_n$. Then:

$$\underbrace{\partial_{s_1=0} \cdots \partial_{s_n=0} \left(\sum_i s_i Y_i\right)^{\otimes n}}_{\text{difference quotients are of desired form} \to \text{also limit}} = \sum_\pi R_\pi Y R_\pi^\dagger$$

$\square$

ok???

e.g. $\partial_{s_1=0} \partial_{s_2=0} (s_1 Y_1 + s_2 Y_2)^{\otimes 2}$

$= \partial_{s_1=0} [Y_2 \otimes (s_1 Y_1 + \cancel{s_2 Y_2}) + (s_1 Y_1 + \cancel{s_2 Y_2}) \otimes Y_2]$

$= Y_2 \otimes Y_1 + Y_1 \otimes Y_2$:)

* Cor: If $[R_\pi, M] = 0 \ \forall \pi$: $M = \bigoplus_k M_k \otimes I_{W_{n,k}}$

* Assume $W_{n,k}$ NOT irreducible: Let

$Q^{(n,k)}$:= projector onto nontrivial invariant subspace

↳ $\bigoplus_{k'} \delta_{kk'} \cdot I_{V_{n,k}} \otimes Q^{(n,k)}$ is S_n-intertwiner,

NOT of form above! ↯

* Assume $W_{n,k_1} \cong W_{n,k_2}$ for $k_1 \neq k_2$: Let

$J: W_{n,k_1} \longrightarrow W_{n,k_2}$ nonzero intertwiner

↳ "$|0\rangle_{V_{n,k_2}} \langle 0|_{V_{n,k_1}} \otimes J$" is S_n-intertwiner

NOT of form above! ↯

not diagonal w.r.t. k!

□