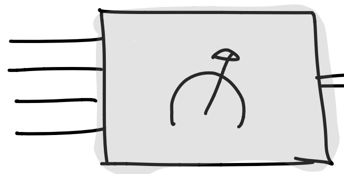
Last week: Compression via typical subspaces $\quad \mathrm{tr}[P_n \rho^{\otimes n}] \to 1$

This week: $\boxed{\rho^{\otimes n} \text{ on } (\mathbb{C}^d)^{\otimes k}}$ quantum IID

Today's motivation: Spectrum estimation

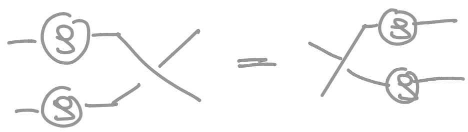
$\rho^{\otimes n}$  $\{\hat{p}_1, \dots, \hat{p}_d\} \approx \{p_1, \dots, p_d\}$

eigenvalues of ρ

Why? ① Half way to mixed state estim. ② Universal Compression ③ Entropy estimation

Symmetries:

* $[R_\pi, \rho^{\otimes n}] = 0 \quad \forall \pi$



"permutation-invariant" state

BUT: NOT only on Sym^n !

$\boxed{\text{PSET}}$

* $U^{\otimes n} \rho^{\otimes n} U^{\dagger \otimes n} = (U \rho U^\dagger)^{\otimes n} \neq \rho^{\otimes n}$ in general

BUT: same eigenvalues

$\hookrightarrow$ Search for measurement w/ both symmetries

$[U^{\otimes n}, P_x] = [R_\pi, P_x] = 0$

WARMUP: $n=2$

$$(\mathbb{C}^d)^{\otimes 2} = \mathrm{Sym}^2(\mathbb{C}^d) \oplus \Lambda^2(\mathbb{C}^d)$$

Measurement $\{P_1 := \Pi_2, \; P_0 := I - \Pi_2\}$:

* Symmetries: $[U^{\otimes 2}, \Pi_2] = 0$ as we well know!!!

  $[R_{1\leftrightarrow 2}, \Pi_2] = 0$ via $\Pi_2 = \frac{1}{2}(I + R_{1\leftrightarrow 2})$

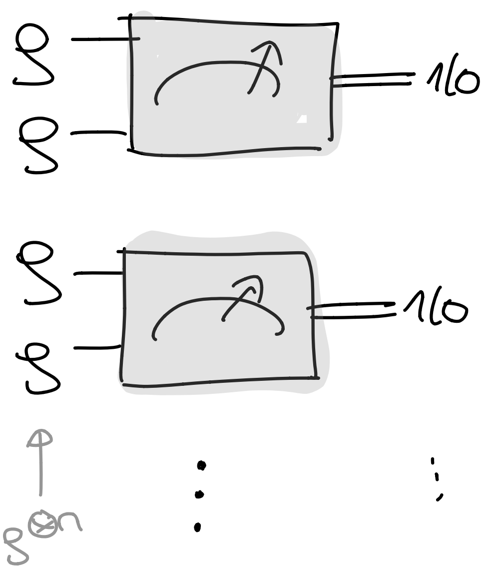
* Do we learn anything?

$$\Pr_{\rho^{\otimes 2}}(\text{outcome } 1) = \mathrm{tr}[\rho^{\otimes 2}\Pi_2] = \frac{1}{2}\left(1 + \mathrm{tr}[\rho^{\otimes 2} R_{1\leftrightarrow 2}]\right)$$

$$\overset{\text{PSET}}{\underset{\text{Swap TRICK}}{=}} \frac{1}{2}\left(1 + \mathrm{tr}[\rho^2]\right) = \frac{1}{2}\left(1 + \sum_{i=1}^{d} p_i^2\right)$$

Some information ☺

* If we repeat N times:



up to $O(\sqrt{1/N})$

$$\frac{\#\text{outcome}=1}{N} \approx \frac{1}{2}\left(1 + \sum_{i=1}^{d} p_i^2\right)$$

"SWAP TEST"

* Qubits: $\{p_i\} = \{p, 1-p\} \to$ complete solution, but:

  breaks S_n symmetry! does not nicely generalize! ↯

Better solution: Decompose according to SU(d)! $\boxed{d=2}$

$$(\mathbb{C}^2)^{\otimes n} \cong \mathrm{Sym}^{k_1}(\mathbb{C}^2)\oplus\dots\oplus \mathrm{Sym}^{k_m}(\mathbb{C}^2)$$

$$\cong \bigoplus_k \underbrace{\mathrm{Sym}^k(\mathbb{C}^2)\oplus\dots\oplus\mathrm{Sym}^k(\mathbb{C}^2)}_{m(n,k)\text{ times}}$$

to be determined

Ok?!?

$$\cong \bigoplus_k \mathrm{Sym}^k(\mathbb{C}^2)\otimes \mathbb{C}^{m(n,k)}$$

i.e.

NOTATION!!!

$$U^{\otimes n} \cong \bigoplus_k T_U^{(k)} \otimes I_{m(n,k)} \quad \text{for all } U\in SU(2)$$

NOTATION!!!

In the last step we used that for any G-rep. $\mathcal{H}$,

$$\overbrace{\mathcal{H}\oplus\dots\oplus \mathcal{H}}^{m\text{ times}} \cong \mathcal{H}\otimes\mathbb{C}^m, \qquad \begin{pmatrix} R_g & & \\ & \ddots & \\ & & R_g\end{pmatrix} \cong R_g\otimes I_m$$

* How about U(2)? $U\in U(2) \longrightarrow \frac{U}{\sqrt{\det U}} \in SU(2)$

↳ $$U^{\otimes n} \cong (\det U)^{n/2}\left(\frac{U}{\sqrt{\det U}}\right)^{\otimes n} = (\det U)^{n/2}\bigoplus_k T^{(k)}_{U/\sqrt{\det U}}\otimes I_{m(n,k)}$$

$$= \bigoplus_k (\det U)^{(n-k)/2}\, T^{(k)}_U \otimes I_{m(n,k)}$$

since $R_U^{(k)}$ homogeneous of degree k

* Define notation for this $U(2)$-rep: $\cong \mathrm{Sym}^k(\mathbb{C}^2)$ as $SU(2)$-rep

$$\begin{cases} V_{n,k} := \mathrm{Sym}^k(\mathbb{C}^2) \\ T_u^{(n,k)} := (\det u)^{(n-k)/2}\, u^{\otimes k} \text{ restricted to } \mathrm{Sym}^k(\mathbb{C}^2) \end{cases}$$

$$\boxed{\begin{aligned} (\mathbb{C}^2)^{\otimes n} &\cong \bigoplus_k V_{n,k} \otimes \mathbb{C}^{m(n,k)} \\ u^{\otimes n} &\cong \bigoplus_k T_u^{(n,k)} \otimes I_{m(n,k)} \end{aligned}}$$

How does this help us? $\rho^{\otimes n} = ?$ measurement for spec. estimation?

* $P_{n,k}$:= projector onto k'th block $\cong \bigoplus_{k'} \delta_{k,k'} \cdot I \otimes I$

↳ Commutes with $u^{\otimes n} \cong \bigoplus_k T_u^{(n,k)} \otimes I$ 😊
(as well as with R_π!) → $\boxed{\text{PSET}}$ $R_\pi \cong \bigoplus_k I \otimes R_\pi^{(n,k)}$

Goal: Analyze measurement $\{P_{n,k}\}$ on $\rho^{\otimes n}$!

$$\Pr_{\rho^{\otimes n}}(\text{outcome } k) = \mathrm{tr}[\rho^{\otimes n} P_{n,k}] = ?$$

* generalizes "swap test" → warmup

* WLOG $\rho = \begin{pmatrix} p & \\ & 1-p \end{pmatrix}$ with $p \geq 1-p$

ρ pure → $k = n$!

* will see that exponent. small unless $\boxed{\hat{p} := \frac{1}{2}\left(1+\frac{k}{n}\right) \approx p}$

How to analyze $\rho^{\otimes n}$? Looks like $U^{\otimes n}$, but...!!

<u>Fact</u>: For every operator X,

$$\boxed{X^{\otimes n} \cong \bigoplus_{k} T_X^{(n,k)} \otimes I_{m(n,k)}}$$

Both LHS+RHS make sense for general X!

<u>Pf</u>: WLOG X invertible. Write as $X = e^{iH}$. Then "=" for $\forall H = H^\dagger$. $\square$

Holomorphic $\to$ "=" $\forall H$.

$$\Rightarrow \Pr_{\rho^{\otimes n}}(\text{outcome } k) \overset{\text{above}}{=} \operatorname{tr}\left[T_\rho^{(n,k)} \otimes I_{m(n,k)}\right]$$

$$= \underbrace{\operatorname{tr}\left[T_\rho^{(n,k)}\right]}_{(2)} \cdot \underbrace{m(n,k)}_{(1)}$$

(1) Recall <u>Clebsch-Gordan rule</u> for $SU(2)$:

$$\mathrm{Sym}^k(\mathbb{C}^2)\otimes\mathbb{C}^2 \cong \begin{cases} \mathrm{Sym}^{k+1}(\mathbb{C}^2)\oplus\mathrm{Sym}^{k-1}(\mathbb{C}^2) & k>0 \\ \mathbb{C}^2 & k=0 \end{cases}$$

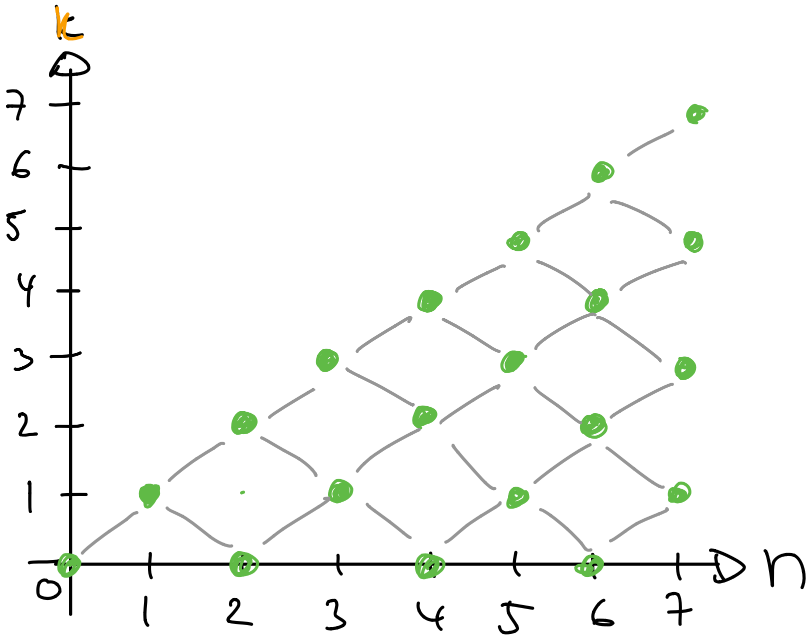
$$\Rightarrow \quad \mathbb{C}^2 = \mathrm{Sym}^1$$

$$(\mathbb{C}^2)^{\otimes 2} = \mathrm{Sym}^1 \otimes \mathbb{C}^2 = \mathrm{Sym}^2 \oplus \mathrm{Sym}^0$$

$m(3,1)=2$

$$(\mathbb{C}^2)^{\otimes 3} = (\mathrm{Sym}^2 \oplus \mathrm{Sym}^0)\otimes\mathbb{C}^2 = \mathrm{Sym}^3 \oplus \boxed{\mathrm{Sym}^1 \oplus \mathrm{Sym}^1}$$

$\vdots$



$$m(n,k) = \#\text{paths from } (0,0) \text{ to } (n,k) \leq \underbrace{\binom{n}{\frac{n+k}{2}}}_{\#\text{ups}} \overset{\text{Coin flip}}{\leq} 2^{n\,h(\hat{p})}$$

where $\hat{p} := \frac{1}{2}\left(1 + \frac{k}{n}\right)$.

(2) $$\operatorname{tr}\left[T_\rho^{(n,k)}\right] = (\det \rho)^{(n-k)/2} \sum_{m=0}^{k} \underbrace{\langle \omega_{m,k-m} | \rho^{\otimes k} | \omega_{m,k-m} \rangle}_{= p^m (1-p)^{k-m} \leq p^k}$$

$$\overset{k\leq n}{\leq} p^{(n-k)/2} (1-p)^{(n-k)/2} \cdot (n+1)\, p^k$$

$$= (n+1)\, p^{(n+k)/2} (1-p)^{(n-k)/2}$$

$$= (n+1)\, 2^{n[\hat{p}\cdot \log(p) + (1-\hat{p})\cdot \log(1-p)]}$$

<u>Together:</u>

$$\operatorname{tr}\left[P_{n,k}\, \rho^{\otimes n}\right] \leq (n+1) \cdot 2^{-n\,\delta(\hat{p}\|p)}$$

Where

$$\boxed{\delta(\hat{p}\|p) := \hat{p}\log\frac{\hat{p}}{p} + (1-\hat{p})\log\frac{1-\hat{p}}{1-p}}$$

BINARY RELATIVE ENTROPY

* another "distance measure" ... but not symmetric
* Pinsker's inequality: $\delta(\hat{p}\|p) \geq \frac{2}{\ln 2}(\hat{p}-p)^2$

PSET

RESULT: If we measure $\{P_{n,k}\}$ on $\rho^{\otimes n}$ then

$$\Pr(|\hat{p}-p| > \varepsilon) \leq (n+1)^2\, 2^{-n\frac{2}{\ln 2}\varepsilon^2}$$

where $\hat{p} := \frac{1}{2}\left(1+\frac{k}{n}\right)$

Keyl & Werner

SUCCESS 😎

$$\sum_{k:|\hat{p}-p|>\varepsilon} \mathrm{tr}[P_{n,k}\rho^{\otimes n}]$$