

## Problem Set 1

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**Problem 1** (Classical and quantum strategies for the GHZ game).

Three players and the referee play the GHZ game, following the same conventions as in class. In particular, the referee chooses each of the four questions  $xyz$  with equal probability  $1/4$ .

- (a) Verify that the winning probability for a general quantum strategy, specified in terms of a state  $|\psi\rangle_{ABC}$  and observables  $A_x, B_y, C_z$ , is given by

$$p_{\text{win,q}} = \frac{1}{2} + \frac{1}{8} \langle \psi_{ABC} | A_0 \otimes B_0 \otimes C_0 - A_1 \otimes B_1 \otimes C_0 - A_1 \otimes B_0 \otimes C_1 - A_0 \otimes B_1 \otimes C_1 | \psi_{ABC} \rangle. \quad (1.1)$$

- (b) Suppose that Alice, Bob, and Charlie play the following randomized classical strategy: When they meet before the game is started, they flip a biased coin. Let  $\pi$  denote the probability that the coin comes up heads. Depending on the outcome of the coin flip, which we denote by  $\lambda \in \{\text{HEADS}, \text{TAILS}\}$ , they use one of two possible deterministic strategies  $a_\lambda(x), b_\lambda(y), c_\lambda(z)$  to play the game. Find a formula analogous to (1.1) for the winning probability  $p_{\text{win,cl}}$  of their strategy.
- (c) In class we argued that even randomized classical strategies cannot do better than  $p_{\text{win,cl}} \leq 3/4$ . Verify this explicitly using the formula you derived in (b).
- (d) Any classical strategy can be realized by a quantum strategy. Show this explicitly for the randomized classical strategy described in (b) by constructing a quantum state  $|\psi\rangle_{ABC}$  and observables  $A_x, B_y, C_z$  such that  $p_{\text{win,cl}} = p_{\text{win,q}}$ .

**Problem 2** (Distinguishing quantum states).

The *trace distance* between two quantum states  $|\phi\rangle$  and  $|\psi\rangle$  is defined by

$$T(\phi, \psi) = \max_{0 \leq Q \leq \mathbb{1}} \langle \phi | Q | \phi \rangle - \langle \psi | Q | \psi \rangle. \quad (1.2)$$

Here,  $0 \leq Q \leq \mathbb{1}$  means that both  $Q$  and  $\mathbb{1} - Q$  are positive semidefinite operators.

- (a) Imagine a quantum source that emits  $|\phi\rangle$  or  $|\psi\rangle$  with probability  $1/2$  each. Show that the optimal probability of identifying the true state by a POVM measurement is given by

$$\frac{1}{2} + \frac{1}{2} T(\phi, \psi).$$

Why can this probability never be smaller than  $1/2$ ?

- (b) Conclude that only orthogonal states (i.e.,  $\langle \phi | \psi \rangle = 0$ ) can be distinguished perfectly.
- (c) Show that the trace distance is a metric. That is, verify that  $T(\phi, \psi) = 0$  if and only if  $|\phi\rangle = e^{i\theta} |\psi\rangle$ , that  $T(\phi, \psi) = T(\psi, \phi)$ , and prove the triangle inequality  $T(\phi, \psi) \leq T(\phi, \chi) + T(\chi, \psi)$ .

You will now derive an explicit formula for the trace distance. For this, consider the spectral decomposition  $\Delta = \sum_i \lambda_i |e_i\rangle\langle e_i|$  of the Hermitian operator  $\Delta = |\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|$ .

- (d) Show that the operator  $Q = \sum_{\lambda_i > 0} |e_i\rangle\langle e_i|$  achieves the maximum in (1.2), and deduce the following formulas for the trace distance:

$$T(\phi, \psi) = \sum_{\lambda_i > 0} \lambda_i = \frac{1}{2} \sum_i |\lambda_i|.$$

- (e) Conclude that the optimal probability of distinguishing the two states in (a) remains unchanged if we restrict to projective measurements.

In class, we used another measure to compare quantum states, namely their overlap  $|\langle\phi|\psi\rangle|$ .

- (f) Show that trace distance and overlap are related by the following formula:

$$T(\phi, \psi) = \sqrt{1 - |\langle\phi|\psi\rangle|^2}.$$

*Hint: Argue that it suffices to verify this formula for two pure states of a qubit, with one of them equal to  $|0\rangle$ , and use the formula derived in part (d).*

This exercise shows that states with overlap close to one are almost indistinguishable by any measurement, justifying our intuition from class.

**Problem 3** (POVMs can outperform projective measurements; Nielsen & Chuang §2.2.6).

Imagine a qubit source that emits either of the two states  $|0\rangle$  and  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  with equal probability  $1/2$ . Your task is to design a measurement scheme that allows to optimally distinguish these two cases. Unfortunately, the states  $|0\rangle$  and  $|+\rangle$  are not orthogonal, so you know that this cannot be done perfectly (e.g., from the previous problem).

Suppose now that your measurement scheme is *not allowed to ever give a wrong answer!* Instead, it is allowed to report one of *three* possible answers: that the true state is  $|0\rangle$ , that the true state is  $|+\rangle$ , or that the measurement outcome is inconclusive. We define the success probability of such a scheme as the probability that you identify the true state correctly.

- (a) Show that for projective measurements the success probability is at most  $1/4$ .  
 (b) Find a POVM measurement that achieves a success probability strictly larger than  $1/4$ .

**Bonus Problem 4** (POVM measurements are physical).

In this exercise, you will show that every POVM measurement can be realized by a projective measurement on a larger system. Thus, let  $\{Q_x\}_{x \in \Omega}$  be an arbitrary POVM measurement on some Hilbert space  $\mathcal{H}_A$ . For simplicity, we will assume that the set of possible outcomes  $\Omega$  is finite.

- (a) Let  $\mathcal{H}_B$  be a Hilbert space with one basis vector  $|x\rangle_B$  for each  $x \in \Omega$ , and fix some arbitrary  $x_0 \in \Omega$ . Show that the linear map

$$|\psi\rangle_A \otimes |x_0\rangle_B \mapsto \sum_x \sqrt{Q_x} |\psi\rangle_A \otimes |x\rangle_B \tag{1.3}$$

is an isometry (an isometry is a map that preserves inner products).<sup>1</sup>

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<sup>1</sup>Every positive semidefinite operator such as  $Q_x$  has a square root  $\sqrt{Q_x}$ , defined by taking the square root of each eigenvalue while keeping the same eigenspaces.

Any isometry from a subspace into a larger Hilbert space can be extended to a unitary operator on the larger space. Thus there exists a unitary  $U_{AB}$  on  $\mathcal{H}_A \otimes \mathcal{H}_B$  that extends the isometry (1.3).

(b) Use  $U_{AB}$  to design a projective measurement  $\{P_{AB,x}\}$  on the joint system  $\mathcal{H}_A \otimes \mathcal{H}_B$  such that

$$Q_x = (\mathbb{1}_A \otimes \langle x_0|_B) P_{AB,x} (\mathbb{1}_A \otimes |x_0\rangle_B)$$

for all outcomes  $x \in \Omega$ .