PHYSICS 491: Symmetry and Quantum Information	May 2, 2017

Entanglement dilution, quantum teleportation, resource inequalities

Lecture 9 — *Michael Walter, Stanford University*

These lecture notes are not proof-read and are offered for your convenience only. They include additional detail and references to supplementary reading material. I would be grateful if you email me about any mistakes and typos that you find.

Last time, we discussed a number of characterizations of the *entanglement entropy* $S_E(\psi) = S(\rho_A) = S(\rho_B)$ of bipartite pure state $|\psi\rangle_{AB}$:

(i) $S_E(\psi)$ describes the optimal quantum compression rate that can be achieved sending over the B-systems of large number of copies of $|\psi\rangle_{AB}$,

(ii) $S_E(\psi)$ is equal to both the distillable entanglement $E_D(\psi)$ and the entanglement cost $E_C(\psi)$, i.e., the rate of ebits that can be obtained from a large number of copies of $|\psi\rangle_{AB}$ and vice versa (with vanishing error as $n \to \infty$):

$$|\psi\rangle_{AB}^{\otimes n} \underset{\longleftarrow}{\overset{\longrightarrow}{\;LOCC\;}} |\Phi_2^+\rangle^{\otimes n S_E(\psi)}$$

For (ii), we wanted to use the chain of inequalities

$$S_E(\psi) \geq E_C(\psi) \geq E_D(\psi) \geq S_E(\psi). \tag{9.1}$$

But we still need to prove the first inequality in eq. (9.1), i.e., that the entanglement cost is at most the entanglement entropy. Moreover, we had claimed without proof that the entanglement entropy is the optimal quantum compression rate in (i). Today, we will discuss both of these results.

9.1 Entanglement dilution

We first consider the task of *entanglement dilution*, where we try to construct many copies of a pure state $|\psi\rangle_{AB}$ from ebits at some rate R:

$$|\Phi_2^+\rangle^{\otimes Rn} \xrightarrow{LOCC} |\psi\rangle_{AB}^{\otimes n}$$

Our idea is follows: Alice can always prepare the entangled state $|\psi\rangle_{AB}^{\otimes n}$ in her laboratory. According to (i), quantum data compression would allow her to transfer the B-systems to Bob at high fidelity by sending roughly $n(S_E(\psi) + \delta)$ qubits. However, sending qubits is disallowed in the current scenario. Can we instead use ebits and LOCC?

It turns out that this is indeed possible. The corresponding protocol is famously known as *quantum teleportation* (Bennett et al., 1993).

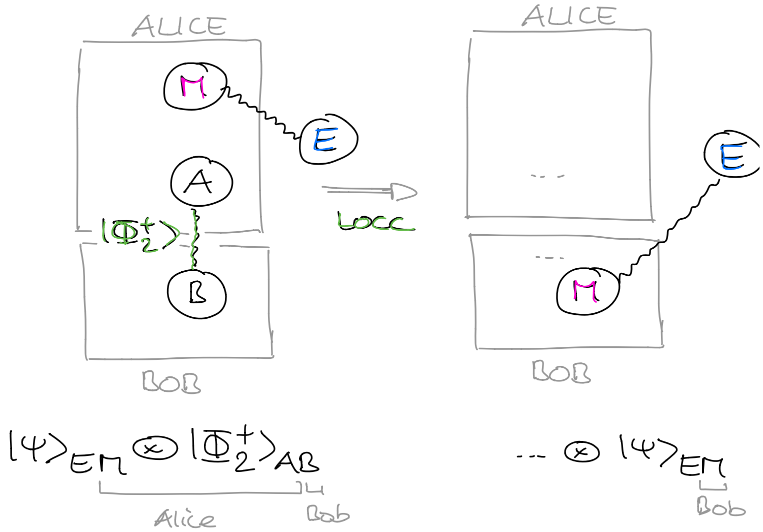

Figure 12: Illustration of the quantum teleportation task: Alice would like to send her M qubit over to Bob, while preserving any entanglement with system E.

9.2 Quantum teleportation

In teleportation, Alice and Bob share an ebit $|\Phi_2^+\rangle_{AB}$ and the goal is for Alice to send an additional qubit M (for "message") that is in her possession over to Bob. We will assume that the qubit M is in a *completely unknown* state and that it might be entangled with some other system, denoted by E (for "environment"). Just as in quantum data compression, we would like to preserve this entanglement. In mathematical terms, what we would like to achieve is the transformation

$$|\psi\rangle_{ME} \otimes |\Phi_2^+\rangle_{AB} \overset{LOCC}{\longrightarrow} |\psi\rangle_{ME},$$

where initially systems AM are in Alice' possession and B in Bob's possession and where we would like to end with M in Bob's possession. See fig. 12 for an illustration.

The *no cloning* theorem suggests that we can only succeed with this task if Alice learns nothing about the state of M. On the other hand, it is clear that she has to apply *some* operation that couples her A and M systems in order to achieve the teleportation task. Since maximally entangled states are locally maximally mixed (problem 2.1), this suggests the following idea: Alice might measure AM in a basis of maximally entangled states, such as

$$\begin{aligned}
|\phi_0\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = (\mathbb{1} \otimes \mathbb{1})\,|\Phi_2^+\rangle, \\
|\phi_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = (\mathbb{1} \otimes Z)\,|\Phi_2^+\rangle, \\
|\phi_2\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = (\mathbb{1} \otimes X)\,|\Phi_2^+\rangle, \\
|\phi_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = (\mathbb{1} \otimes XZ)\,|\Phi_2^+\rangle,
\end{aligned} \tag{9.2}$$

which we may summarize by $|\phi_k\rangle = (\mathbb{1} \otimes U_k)\,|\Phi_2^+\rangle$. When she performs the projective measurement

$P_{AM,k} = |\phi_k\rangle\langle\phi_k|_{AM}$,

$$\begin{aligned}
\Pr(\text{outcome } k) &= (\langle\psi|_{ME} \otimes \langle\Phi_2^+|_{AB}) \left(P_{AM,k} \otimes \mathbb{1}_{EB}\right) (|\psi\rangle_{ME} \otimes |\Phi_2^+\rangle_{AB}) \\
&= \operatorname{tr}\left[P_{AM,k} \operatorname{tr}_{EB}\left[|\psi\rangle\langle\psi|_{ME} \otimes |\Phi_2^+\rangle\langle\Phi_2^+|_{AB}\right]\right] \\
&= \operatorname{tr}\left[P_{AM,k}\left(\operatorname{tr}_E\left[|\psi\rangle\langle\psi|_{ME}\right] \otimes \frac{\mathbb{1}_A}{2}\right)\right] \\
&= \frac{1}{2}\operatorname{tr}\left[|\phi_k\rangle\langle\phi_k|_{AM} \left(\operatorname{tr}_E\left[|\psi\rangle\langle\psi|_{ME}\right] \otimes \mathbb{1}_A\right)\right] \\
&= \frac{1}{2}\operatorname{tr}\left[\frac{\mathbb{1}_M}{2} \operatorname{tr}_E\left[|\psi\rangle\langle\psi|_{ME}\right]\right] \\
&= \frac{1}{4}\operatorname{tr}\left[|\psi\rangle\langle\psi|_{ME}\right] = \frac{1}{4}.
\end{aligned}$$

Thus her measurement outcome is completely random and uninformative, as desired. If the outcome is k, what is the corresponding post-measurement state on ME? It is given by

$$\begin{aligned}
&2\left(\langle\phi_k|_{AM} \otimes \mathbb{1}_{EB}\right)\left(|\psi\rangle_{ME} \otimes |\Phi_2^+\rangle_{AB}\right) \\
&= 2\left(\langle\phi_k|_{AM} \otimes \mathbb{1}_{EB}\right)\left(\mathbb{1}_{ME} \otimes |\Phi_2^+\rangle_{AB}\right)|\psi\rangle_{ME} \\
&= 2\left(\langle\Phi_2^+|_{AM} \otimes \mathbb{1}_{EB}\right)\left(\mathbb{1}_{ME} \otimes |\Phi_2^+\rangle_{AB}\right)\left(U_{M,k}^\dagger \otimes \mathbb{1}_E\right)|\psi\rangle_{ME} \\
&= 2\left(\mathbb{1}_E \otimes \underbrace{\left(\left(\langle\Phi_2^+|_{AM} \otimes \mathbb{1}_B\right)\left(\mathbb{1}_M \otimes |\Phi_2^+\rangle_{AB}\right)\right)}_{=?}\right)\left(U_{M,k}^\dagger \otimes \mathbb{1}_E\right)|\psi\rangle_{ME}.
\end{aligned}$$

Let's calculate the indicated term directly from its definition:

$$\begin{aligned}
&\left(\langle\Phi_2^+|_{AM} \otimes \mathbb{1}_B\right)\left(\mathbb{1}_M \otimes |\Phi_2^+\rangle_{AB}\right) \\
&= \frac{1}{2}\sum_{x,y}\left(\langle x|_A \otimes \langle x|_M \otimes \mathbb{1}_B\right)\left(|y\rangle_A \otimes \mathbb{1}_M \otimes |y\rangle_B\right) \\
&= \frac{1}{2}\sum_{x,y}\langle x|y\rangle\,|y\rangle_B\langle x|_M = \frac{1}{2}\sum_x |x\rangle_B\langle x|_M
\end{aligned}$$

Remarkably, this is nothing but the identity map from two qubit M to B (up to an overall factor 1/2)! As a direct consequence, we obtain that the post-measurement state is given by

$$\begin{aligned}
&2\left(\mathbb{1}_E \otimes \underbrace{\left(\left(\langle\Phi_2^+|_{AM} \otimes \mathbb{1}_B\right)\left(\mathbb{1}_M \otimes |\Phi_2^+\rangle_{AB}\right)\right)}_{=?}\right)\left(U_{M,k}^\dagger \otimes \mathbb{1}_E\right)|\psi\rangle_{ME} \\
&= \left(\mathbb{1}_E \otimes \sum_x |x\rangle_B\langle x|_M\right)\left(U_{M,k}^\dagger \otimes \mathbb{1}_E\right)|\psi\rangle_{ME} = \left(U_{B,k}^\dagger \otimes \mathbb{1}_E\right)|\psi\rangle_{BE},
\end{aligned}$$

where we write $|\psi\rangle_{BE}$ for the same state as $|\psi\rangle_{ME}$ but now living in the two-qubit Hilbert space corresponding to systems BE rather than ME. If Alice sends over $k \in \{0, 1, 2, 3\}$, which requires *two bits of classical communication*, then Bob can apply the unitary $U_{B,k}$ on his system. Thus, our two protagonists have produced the state $|\psi\rangle_{BE}$ (or $|\phi_k\rangle_{AM} \otimes |\psi\rangle_{BE}$, if we are interested in the state

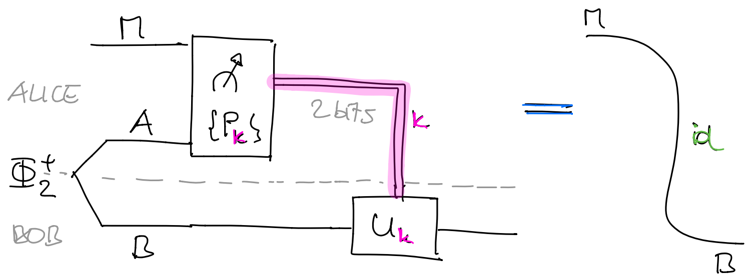

Figure 13: Quantum teleportation as a quantum circuit.

of all four quantum systems). This concludes the teleportation protocol – we have successfully sent over Alice' M system to Bob while preserving all entanglement with E.

See fig. 13 for an illustration of the teleportation procedure in quantum circuit notation. The notation will be clear to you, but next time, in lecture 10, we will give a more systematic introduction to quantum circuits. Note that quantum teleportation is indeed an LOCC protocol – we only applied local operations and Alice needed to send over 2 bits of classical communication. We emphasize that no asymptotics was required and the teleportation procedure worked perfectly, without disturbing the sent-over state at all. Moreover, it is *composable* in the sense that we can send over a state of N qubits by using N ebits (and $2N$ bits of classical communication).

From quantum compression to entanglement dilution

In particular, we can use this to convert any quantum data compression protocol into a entanglement dilution protocol at the same rate: Alice simply prepares the target state $|\psi\rangle_{AB}^{\otimes n}$ in her laboratory and then applies the data compression protocol, with quantum communication replaced by ebits and LOCC. In particular, this is true for an optimal quantum compression protocol. It follows that

$$S_E(\psi) \geq R_{\text{compr}}^{\text{opt}}(\psi) \geq E_C(\psi) \geq E_D(\psi) \geq S_E(\psi)$$

where we denote by $R_{\text{compr}}^{\text{opt}}(\psi)$ the optimal quantum compression rate for many copies of $|\psi\rangle_{AB}$. The first inequality holds because we know from lecture 7 that we can compress at rate $S_E(\psi)$; the second inequality holds by what we just discussed; and the remaining inequalities we had already justified last time. As a consequence,

$$S_E(\psi) = R_{\text{compr}}^{\text{opt}}(\psi) = E_C(\psi) = E_D(\psi).$$

We have thus proved *both* outstanding claims in points (i) and (ii) mentioned at the beginning of today's lecture.

Entanglement swapping

Teleportation can also be used to establish entanglement between distant parties. For example, suppose that Alice and Bob are completely uncorrelated but that each of them shares an ebit with an intermediate party, Charlie, as displayed in fig. 14. Charlie and Bob can use their ebit $|\Phi_2^+\rangle_{C_2B}$ to teleport over Charlie's C_1 system to Bob. The result is a maximally entangled state $|\Phi_2^+\rangle_{AB}$

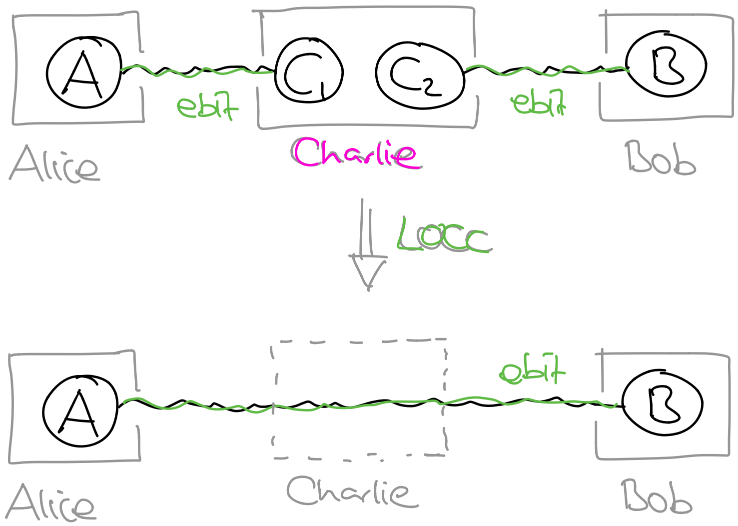

Figure 14: Entanglement swapping establishes entanglement by using quantum teleportation through intermediate parties (for simplicity, only a single intermediate party is displayed).

between Alice and Bob. (Here, we crucially used the fact that teleportation preserves the pre-existing entanglement between C_1 and A.)

The very same idea works if we have many intermediate parties. By successive teleportation, we can establish long-range entanglement between Alice and Bob. This protocol is known as *entanglement swapping*.

9.3 Resource inequalities

We have seen that it can be quite useful to compare different information processing resources with each other. In quantum information theory we like to use a formal notation for this. For example, we would write teleportation as a *resource inequality*

$$\text{ebit} + 2[c \to c] \geq [q \to q]. \tag{9.3}$$

This inequality means that an ebit and 2 bits of classical communication ($[c \to c]$) can be used to send one qubit of quantum communication ($[q \to q]$). Sometimes, ebits are also denoted by $[qq]$.

What other resource inequalities do we know? Clearly,

$$[q \to q] \geq \text{ebit},$$

since we can always prepare the ebit at Alice' side and send over half of it to Bob. However, ebit $\ngeq [q \to q]$, since entanglement alone cannot be used to communicate.

Another example is

$$[q \to q] \geq [c \to c],$$

since Alice can encode a classical bit x into the state $|x\rangle$ of a qubit, send that qubit over, and have Bob measure $\{|x\rangle\langle x|\}$. However, $[q \to q] \ngeq 2[c \to c]$. This is a consequence of the *Holevo bound*, but we have not had time to discuss this in class.

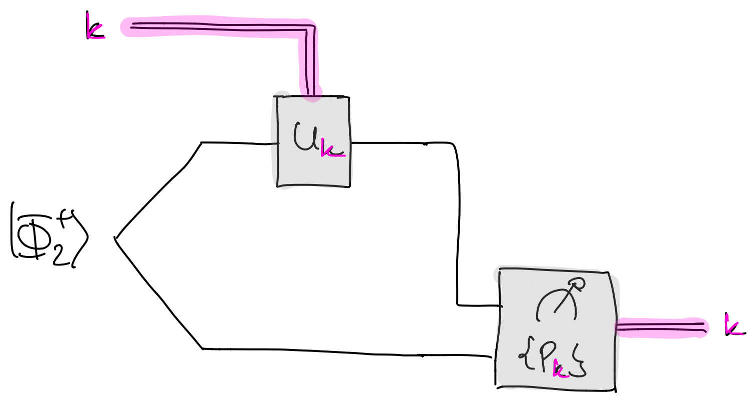

Figure 15: In superdense coding, Alice can communicate two classical bits to Bob by sending over a single qubit that is part of a shared ebit.

Superdense coding

What is in fact possible, though, is to send over 2 classical bits by sending a qubit if we can use some entanglement:

$$\text{ebit} + [q \to q] \geq 2[c \to c]. \tag{9.4}$$

We can think of this as an analogue or "dual" of teleportation. However, it is *not* a converse, since both protocols use ebits as a resource. By combining eqs. (9.3) and (9.4), we find that

$$[q \to q] \equiv 2[c \to c] \quad (\text{mod ebit}),$$

although this is not a very standard notation.

How can we achieve eq. (9.4)? The corresponding protocol is known as *superdense coding*, and it is in fact very simple: Suppose that Alice and Bob share an ebit $|\Phi_2^+\rangle_{AB}$. Alice first applies one out of the four unitaries U_k to her qubit before sending it over to Bob. But now Bob has one of the four states $|\phi_k\rangle$ in his possession. Since they are orthogonal, he can simply perform the projective measurement $P_k = |\phi_k\rangle\langle\phi_k|$ to perfectly distinguish the four states and thereby recover k. In this way, Alice can send over an arbitrary message $k \in \{0, \ldots, 3\}$ to Bob, amounting to two bits of classical communication. See fig. 15 for an illustration.

A glance at quantum channels

At this point, it would be natural to introduce *quantum channels* which are described mathematically by so-called completely positive, trace-preserving maps. They provide a unified framework for modelling general quantum information processing protocols. In this course, we only had time for a brief discussion at the end of today's lecture, but you are encouraged to have a look at, e.g., Wilde (2013).

Bibliography

Charles H Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, *Physical Review Letters*, 70(13):1895, page 1895, 1993.

Mark M Wilde. *Quantum information theory.* Cambridge University Press, 2013.