PHYSICS 491: Symmetry and Quantum Information	April 27, 2017
Compression and entanglement, entanglement transformations	
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These lecture notes are not proof-read and are offered for your convenience only. They include additional detail and references to supplementary reading material. I would be grateful if you email me about any mistakes and typos that you find.

Today we will discuss some entanglement theory of bipartite pure states (i.e., pure states $|\psi\rangle_{AB}$ with two subsystems). First, we will solve the problem of compressing subsystems of entangled states. Then we study transformations between pure states in order to compare them in their entanglement.

8.1 Compression and entanglement

Density operator do not only arise when describing statistical ensembles, but also when describing subsystems of entangled states. This suggests a second kind of quantum compression task (Schumacher, 1995): Given many copies of a bipartite pure state, $|\psi\rangle_{AB}^{\otimes n}$, we would like to send over the B-systems to Bob by first compressing the B-systems, sending over a minimal number of qubits, and decompressing at Bob's side (fig. 11). Thus, if $|\tilde{\psi}\rangle_{A^n B^n}$ is the state after compression and decompression, we would like that

$$|\tilde{\psi}\rangle_{A^n B^n} \approx |\psi\rangle_{AB}^{\otimes n}$$

(say, on average).

We can achieve this using the same protocol as before – but this time applied to the B-systems only. Let us accordingly write $\tilde{P}_{B^n}$ for the typical projector defined in terms of the eigenvalues $\{p, 1-p\}$ of $\rho_B := \operatorname{tr}_A [|\psi\rangle\langle\psi|_{AB}]$, and $\tilde{\mathcal{H}}_{B^n} \subseteq (\mathbb{C}^2)^{\otimes n}$. Then the protocol reads as follows:

- Measure the observable $\tilde{P}_{B^n}$.
- If the outcome is 1, then the post-measurement state lives in $(\mathbb{C}^2)^{\otimes n} \otimes \tilde{\mathcal{H}}_{B^n}$. We send over the B-systems using roughly $n(S(\rho) + \delta)$ qubits.
- If the outcome is 0, send over some arbitrary state (or simply fail).

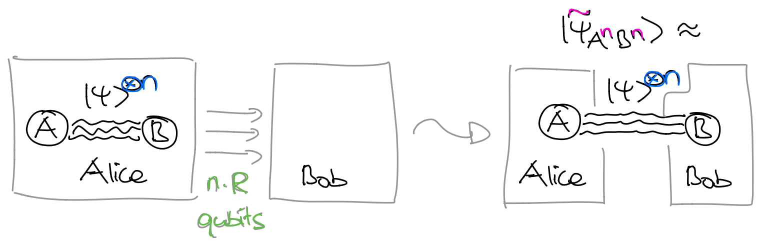

Figure 11: Alice wants to send half of her entangled states $|\psi\rangle_{AB}^{\otimes n}$ over to Bob at transmission rate R.

The probability that the measurement $\tilde{P}_{B^n}$ yields outcome 1 is given by

$$q := \langle \psi_{AB}^{\otimes n} | \mathbb{1}_{A^n} \otimes \tilde{P}_{B^n} | \psi_{AB}^{\otimes n} \rangle = \operatorname{tr}\left[\rho_B^{\otimes n} \tilde{P}_{B^n} \right] \to 1.$$

In this case, the post-measurement state is

$$\frac{(\mathbb{1}_{A^n} \otimes \tilde{P}_{B^n}) |\psi_{AB}\rangle^{\otimes n}}{\sqrt{q}}$$

and its squared overlap with the original state is

$$\frac{1}{q} |\langle \psi_{AB}^{\otimes n} | \mathbb{1}_{A^n} \otimes \tilde{P}_{B^n} | \psi_{AB}^{\otimes n} \rangle|^2 = \frac{q^2}{q} = q \to 1.$$

It follows that the average overlap is at least

$$E |\langle \psi_{AB}^{\otimes n} | \tilde{\psi}_{A^n B^n} \rangle|^2 \geq q^2 \to 1.$$

Thus we have solved the problem of sending over half of an entangled state: Our compression protocol works at an asymptotic rate of $S(\rho) + \delta$ qubits. Again, it turns out that this rate is optimal – we will be able to prove this next time in lecture 9.

We thus obtain a second operational interpretation of the von Neumann entropy: When applied to the reduced density matrix ρ_B of a bipartite pure state, it is the minimal rate of qubits required to send over the B-systems of many copies of the state from Alice to Bob. This is very intuitive and in line with our discussions in section 3.2 and problem 2.1: For pure states, the mixedness of the reduced density operators is a signature of entanglement. The more entanglement there is in $|\psi\rangle_{AB}$ the more qubits we need to send over to Bob in order to create this state between Alice and Bob. This gives a good justification why in the literature the expression

$$S_E(\psi_{AB}) = S(\rho_A) = S(\rho_B) \tag{8.1}$$

is often called the *entanglement entropy* of the bipartite pure state $|\psi\rangle_{AB}$.

Example. *If $|\psi\rangle_{AB} = |0\rangle_A \otimes |0\rangle_B$ then we do not need to send any quantum information – we can simply prepare the state $|0\rangle$ on Bob's end. If $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|00\rangle_{AB} + |11\rangle_{AB})$ is a maximally entangled state then we cannot compress the B-systems at all and need to send a rate of $S_E = 1$.*

The task that we just solved could be more aptly called "quantum state transfer", since we seek to transfer the state of the B-systems over to Bob while preserving all correlations with the purifying A-systems (sadly, this term is usually used with a different connotation). It is a special case of the more general problem of *quantum state merging*, where the receiver already possesses part of the state – we will have a peek at this next week.

Remark. *Again, note that our protocol only depended on the eigenvalues of ρ_B (equivalently, of ρ_A). The same modification discussed in problem 3.3 allows us to build a universal protocol at fixed rate S_0 that works for all states whose entanglement entropy is bounded by $S_E < S_0$.*

Remark. *It is possible to show that the task of sending over half of a maximally entangled state at minimal qubit cost is a more difficult problem than the compression of quantum sources in the sense that whenever we have a protocol for the former we can use it to compress arbitrary quantum sources with associated density operator ρ_B.*

8.2 Entanglement transformations

Let us talk some more about entanglement. For pure states, $|\psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B$ means that the state is entangled. But how can be compare and quantify different states in their entanglement? One approach is to assign to each state some arbitrary numbers that we believe reflect aspects of their entanglement properties – e.g., the entanglement entropy S_E from eq. (8.1), the Rényi entropy from problem 2.1, or simply the collection of all eigenvalues of ρ_A or ρ_B. Yet, this might seem somewhat ad hoc and so is not completely satisfactory.

A more operational approach would be to compare two states $|\phi\rangle_{AB}$ and $|\psi\rangle_{AB}$ by studying whether one can be transformed into the other: What family of operations should we consider in such a transformation? Since our goal is compare entanglement, we should only allow for operations that cannot create entanglement from unentangled states. We already briefly mentioned such a family when we discussed mixed-state entanglement in section 4.1: It is LOCC, short for *Local Operations and Classical Communication*. Here, we imagine that Alice and Bob each have their separate laboratory.

- Local operations refers to arbitrary quantum operations that can be done on Alice' and Bob's subsystems. We allow any combination of unitaries, adding auxiliary systems, performing partial traces, and measurements.
- Classical communication refers to Alice and Bob's ability to exchange measurement outcomes. Thus, Bob's local operations can depend on Alice's previous measurement outcomes, and vice versa.

Thus we are interested to study whether

$$|\psi\rangle_{AB} \xrightarrow{LOCC} |\phi\rangle_{AB}.$$

If yes, then we could say that $|\psi\rangle_{AB}$ is at least as entangled as $|\phi\rangle_{AB}$ – indeed, the former is as useful as the latter for any nonlocal quantum information processing task, since we can always convert $|\psi\rangle_{AB}$ into $|\phi\rangle_{AB}$ when required.

Remark. *Note that the setup here is very different from quantum data compression – there, we wanted to minimize the amount of quantum communication sent. Here, we do* not *allow* any *quantum communication.*

Example 8.1. *Consider the* EPR pair *or* ebit $|\Phi_2^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, *as well as its generalization, the* maximally entangled state *in d-dimensions*

$$|\Phi_d^+\rangle := \frac{1}{\sqrt{d}} \sum_i |ii\rangle .$$

It is intuitive and also true that

$$|\Phi_d^+\rangle \xrightarrow{LOCC} |\Phi_{d'}^+\rangle$$

if and only if $d \geq d'$. The "if" is only obvious if $d = 2^n$ and $d' = 2^{n'}$, since in this case the transformation can simply be achieved by tracing out $n - n'$ of the qubit. For the "only if", one can argue that the number of terms in the Schmidt decomposition, which is d for $|\Phi_d^+\rangle$, can never increase under LOCC. We will not prove this in class, but you may verify both claims in problem 4.2.

However, it might be instructive to see concretely how the conversion $|\Phi_3^+\rangle \to |\Phi_2^+\rangle$ *can be achieved, since the general case can be proved completely analogously. The trick is to note that, while*

$$|\Phi_3^+\rangle = \frac{1}{\sqrt{3}} \left(|11\rangle + |22\rangle + |33\rangle\right),$$

we can also write

$$|\Phi_2^+\rangle = (\mathbb{1}_A \otimes U_B) \frac{1}{\sqrt{3}} \left(|\psi_1\rangle |1\rangle + |\psi_2\rangle |2\rangle + |\psi_3\rangle |3\rangle\right) \tag{8.2}$$

where U_B *is some unitary on* B*. Here, the* $|\psi_i\rangle \in \mathbb{C}^2$ *are normalized but non-orthogonal states such that* $\frac{1}{3}\sum_i |\psi_i\rangle\langle\psi_i| = \frac{1}{2}\sum_{i=1}^2 |i\rangle\langle i|$ *(!). For example, you can use the three states constructed in example 2.1.*

Alice and Bob can now apply the following LOCC protocol: First, Alice applies the isometry

$$|i\rangle_A \mapsto |\phi_i\rangle_A \otimes \frac{1}{\sqrt{3}} \sum_j \omega^{ij} |j\rangle_{A'},$$

where $\omega = e^{2\pi i/3}$ *is a primitive third root of unity (as in problem 1.4, this can be realized by adding an auxiliary system and performing a unitary). The second system is necessary to ensure that this is indeed an isometry (recall that the* $|\phi_i\rangle_A$ *alone are not orthogonal). When applied to* $|\Phi_3^+\rangle$*, the resulting state is*

$$\frac{1}{3} \sum_{i,j} \omega^{ij} |\phi_i\rangle_A \otimes |j\rangle_{A'} \otimes |i\rangle_B .$$

Alice now measures her auxiliary A' *system in the standard basis. The probability of each outcome is* $1/3$*. After discarding* A'*, the corresponding post-measurement state si*

$$\frac{1}{\sqrt{3}} \sum_i \omega^{ij} |\phi_i\rangle_A \otimes |i\rangle_B .$$

This almost looks as desired – except for the phases. To get rid of them, Alice sends j *over to Bob, and Bob applies the diagonal unitary* $|i\rangle_B \mapsto \omega^{-ij} |i\rangle_B$*. We obtain*

$$\frac{1}{\sqrt{3}} \sum_i |\phi_i\rangle_A \otimes |i\rangle_B .$$

At last, Bob applies to unitary U_B*. Thus, Alice and Bob have obtained eq.* (8.2) *– done!*

The theory of exact interconversion is solved for bipartite pure states. However, there are many parameters – the entire spectrum of ρ_A and ρ_B matters (Nielsen, 1999, Nielsen and Vidal, 2001). It turns out that the asymptotic theory simplifies tremendouly, and we will discuss this now. The key idea is that instead of converting many copies of two arbitrary states into each other, we will study the conversion into (and from) a common resource or "currency" of entanglement. This common resource is the *maximally entangled state* or *ebit* $|\Phi_2^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

8.3 Entanglement concentration

The first problem that we want to study is the following: Given many copies of a state $|\psi\rangle_{AB}$, convert them by LOCC into as many ebits as possible:

$$|\psi\rangle_{AB}^{\otimes n} \xrightarrow{LOCC} \approx |\Phi_2^+\rangle^{\otimes Rn}$$

Just as in the case of data compression, we are interested in the maximal rate R that can be achieved with error going to zero for $n \to \infty$ (or rather its supremum). This is called the *distillable entanglement* $E_D(\psi)$ of the state $|\psi\rangle_{AB}$.

For example, $E_D(|\phi^+\rangle) = 1$ and, more generally, $E_D(|\Phi_d^+\rangle) = \log d$ (cf. example 8.1). Instead of proving directly, we will consider the general case right away.

We will approach this problem by first focusing on Alice' Hilbert space,

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_{n,j}^A \otimes W_{n,j}^A,$$

where the superscripts indicate that we refer to Alice. If we write $\rho_A = \operatorname{tr}_B [|\psi\rangle\langle\psi|_{AB}]$, then

$$\rho_A^{\otimes n} \cong \bigoplus_j T_{\rho_A}^{(n,j)} \otimes \mathbb{1}_{W_{n,j}^A} = \bigoplus_j p_j \rho_{V_{n,j}^A} \otimes \tau_{W_{n,j}^A}.$$

On the right-hand side, we have written each direct summand as a probability (p_j) times a tensor product of density operators – this is possible since the direct summands are positive semidefinite. Note that p_j is nothing but the probability of obtaining outcome j when measuring P_j on Alice's qubits, and recall that $\tau_{W_{n,j}^A} = \mathbb{1}_{W_{n,j}^A}/m(n,j)$ was our notation for a maximally mixed state. Now suppose that Alice does indeed perform the measurement P_j on her qubits and receives outcome j. Then her post-measurement state is $\rho_{V_{n,j}^A} \otimes \tau_{W_{n,j}^A}$. What does the overall post-measurement state look like? Let us first guess a purification. We can purify $\rho_{V_{n,j}^A}$ to some arbitrary $|\tilde{\psi}\rangle_{V_{n,j}^A V_{n,j}^B}$, and $\tau_{W_{n,j}^A}$ to the maximally entangled state $|\Phi^+\rangle_{W_{n,j}^A W_{n,j}^B}$. Hence, a purification of her post-measurement states looks like

$$\begin{aligned}|\tilde{\psi}\rangle_{V_{n,j}^A V_{n,j}^B} \otimes |\Phi^+\rangle_{W_{n,j}^A W_{n,j}^B} &\in (V_{n,j}^A \otimes V_{n,j}^B) \otimes (W_{n,j}^A \otimes W_{n,j}^B) \\ &\cong (V_{n,j}^A \otimes W_{n,j}^A) \otimes (V_{n,j}^B \otimes W_{n,j}^B) \subseteq (\mathbb{C}^2)^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n}.\end{aligned}$$

We now use an important result that we have not met before: *Any two purifications of a quantum state are related by a unitary on the auxiliary Hilbert space.* In the present context, this means that the post-measurement state is precisely equal to

$$(\mathbb{1}_{A^n} \otimes U_{B^n}) \left(|\tilde{\psi}\rangle_{V_{n,j}^A V_{n,j}^B} \otimes |\Phi^+\rangle_{W_{n,j}^A W_{n,j}^B} \right),$$

where U_{B^n} is some unitary acting on Bob's Hilbert space. If Bob applies $U_{B^n}^\dagger$ and both parties discard their $V_{n,j}$-systems, they arrive at the maximally entangled state

$$|\Phi^+\rangle_{W_{n,j}^A W_{n,j}^B}.$$

But with high probability, j will be such this is a maximally entangled state of dimension no smaller than $2^{n(S(\rho_A)-\delta)}$. According to example 8.1, we can convert this into $\lfloor n(S(\rho_A)-\delta)\rfloor$ ebits.

Thus we find that using the preceding entanglement concentration protocol, which is completely universal, we can distill entanglement at rates arbitrary close to the entanglement entropy $S_E(\psi) = S(\rho_A)$. In other words:

$$E_D(\psi) \geq S_E(\psi)$$

Remark. *Since* $|\psi\rangle_{AB}^{\otimes n}$ *is in the symmetric subspace* $\mathrm{Sym}^n(\mathbb{C}^2 \otimes \mathbb{C}^2)$*, we can identify the maximally entangled state much more precisely by using representation theory. This avoids the need to appeal to Uhlmann's theorem and makes the protocol quite a bit more concrete. You are welcome pursue this idea in problem 4.3.*

Is this rate optimal? Yes – we will show this next time using a "thermodynamics argument".

- First, we will study the reverse transformation (i.e., from perfects ebits to copies of $|\psi\rangle_{AB}$). We will show that the minimal rate of ebits required, known as the *entanglement cost* $E_C(\psi)$, is no more than $S_E(\psi)$.
- We can thus consider the "cyclic process" starting and ending at ebits:

$$\phi_+^{\otimes E_C(\psi)n} \to \psi^{\otimes n} \to \phi_+^{\otimes E_D(n)}$$

- Necessarily, $E_C(\psi) \geq E_D(\psi)$, because otherwise we could create ebits from nothing! Why is this not possible? See example 8.1 above for the exact case; the approximate case follows similarly by tracking epsilons and deltas, as you may show in problem 4.2.
- By combing all results, we will find that $S_E(\psi) \geq E_C(\psi) \geq E_D(\psi) \geq S_E(\psi)$ so they are all equal:

$$S_E(\psi) = E_C(\psi) = E_D(\psi)$$

This is the main result of the bipartite entanglement of pure states, and it gives us two new operational interpretations of the von Neumann entropy which justify its use as an entanglemen measure for pure states: The von Neumann entropy measures the maximal rate at which ebits can distilled from many copies of a state $|\psi\rangle_{AB}$, as well as the minimal rate of ebits required to produce many copies of $|\psi\rangle_{AB}$ (up to arbitrarily high fidelity).

More generally, if we have two states $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$ then we can convert the former into the latter by LOCC at optimal rate $S_E(\psi)/S_E(\phi)$ – this is a satisfyingly simple resolution of the question that we set out to solve.

Discussion

Let us close with two remarks. First, the approach that we pursued above to study entanglement transformations was rooted in the idea of the ebit as a *resource*. This idea of setting up *resource theories* to compare different quantum states in their relative strength for certain tasks has been quite fruitful in quantum information theory, and there are many further examples (e.g., in quantum thermodynamics).

Second, you might wonder how the above story generalizes to mixed states ρ_{AB}. It turns out that in this case the entanglement theory is much more complicated. We already saw hints of this in section 4.1 when we discussed that even deciding whether a given state ρ_{AB} is separable is in general an NP-hard problem. In addition, while the same definitions can be made as above, there are many new phenomena. For example, in general we have that $E_C(\rho) > E_D(\rho)$, meaning that the conversion via ebits is in general asymptotically irreversible! In fact, there are entangled mixed states states such that $E_C(\rho) > 0$ while $E_D(\rho) = 0$. We call them *bound entangled states* – these states are entangled but no ebits can be distilled from them at a positive rate.

Relatedly (because every mixed state ρ_{AB} can be purified to a tripartite pure states $|\psi\rangle_{ABC}$) the entanglement of pure states with more than two subsystems is similarly complicated.

Bibliography

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