Last time: Spectrum estimation $\boxed{d=2}$

$$\rho^{\otimes n} \xrightarrow[???]{\text{meas.}} \text{eigenvalues } p \geq 1-p \text{ of } \rho$$

Approach:

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_j \otimes \mathbb{C}^{m(n,j)}$$

P_j = projection onto spin-j block

$\hookrightarrow$ $\boxed{\Pr(\text{outcome } j) = \operatorname{tr}[P_j \rho^{\otimes n}] = ?}$

We will show that exponentially small unless $\hat{p} = \frac{1}{2} + \frac{j}{n} \approx p$.

Idea:

$$\rho^{\otimes n} \cong \bigoplus_j \underbrace{\det(\rho)^{n/2} T^{(j)}_{\rho/\sqrt{\det(\rho)}}}_{=: T^{(n,j)}_\rho} \otimes I$$

$\tilde{\rho} \in SL$

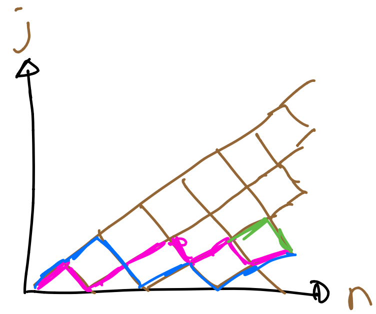
$\leftarrow$ repr. of GL !!!

$$\Longrightarrow \operatorname{tr}[P_j \rho^{\otimes n}] = \operatorname{tr}\left[T^{(n,j)}_\rho\right] \cdot m(n,j)$$

"①" LAST TIME ②

② Clebsch–Gordan rule:

$$V_j \otimes V_{\frac{1}{2}} = \begin{cases} V_{j+\frac{1}{2}} \oplus V_{j-\frac{1}{2}} & (j>0) \\ V_{j+\frac{1}{2}} & (j=0) \end{cases}$$



$$m(n,j) = \#\text{paths from } (0,0) \text{ to } (n,j) \rightarrow \boxed{\text{Pset}}$$

$$\leq \binom{n}{\frac{n}{2}+j}$$ $\frac{n}{2}+j$ ups to end at j

$$\leq 2^{n\,h\left(\underbrace{\frac{1}{2}+\frac{j}{n}}_{\hat{p}}\right)}$$

where $h(\hat{p}) = -\hat{p}\log\hat{p} - (1-\hat{p})\log(1-\hat{p})$ BINARY ENTROPY

① $$\mathrm{tr}\left[T_{\rho}^{(n,j)}\right] \leq (2j+1)\, p^{\frac{n}{2}+j}(1-p)^{\frac{n}{2}-j}$$ $\geq p^{\frac{n}{2}+j}(1-p)^{\frac{n}{2}-j}$

$$= (2j+1)\, 2^{n\cdot\left[\hat{p}\log p + (1-\hat{p})\log(1-p)\right]}$$

$$= (2j+1)\, 2^{-n\left[\hat{h}(\hat{p}) + \delta(\hat{p}\|p)\right]}$$

where

$$\delta(\hat{p}\|p) = \hat{p}\log\frac{\hat{p}}{p} + (1-\hat{p})\log\frac{1-\hat{p}}{1-p}$$ BINARY RELATIVE ENTROPY

- another "distance measure" but not symmetric
- $S(\hat{p} \| p) \geq \frac{2}{\ln 2} (\hat{p}-p)^2$ PINSKER $\longrightarrow$ $\boxed{\text{Pset}}$

Success!

$$\mathrm{tr}[P_j \rho^{\otimes n}] \leq (2j+1)\, 2^{-n S(\hat{p} \| p)}$$

is exponentially small unless $\hat{p} = \frac{1}{2} + \frac{j}{n} \approx p$.

$$\Longrightarrow \Pr(|\hat{p}-p| > \varepsilon) = \sum_{|\hat{p}-p|>\varepsilon} \mathrm{tr}[P_j \rho^{\otimes n}]$$

Pinsker

$$\leq (n+1)^2\, 2^{-\frac{2}{\ln 2} n \varepsilon^2} \sim 2^{-\frac{2}{\ln 2} n \varepsilon^2}$$

Another interpretation:

$$\tilde{P} = \sum_{|\hat{p}-p| \leq \varepsilon} P_j \qquad \mathrm{tr}[\tilde{P} \rho^{\otimes n}] \approx 1$$

$\Longrightarrow$ $\rho^{\otimes n} \approx \boxed{\tilde{P} \rho^{\otimes n} \tilde{P}}$ lives mostly on "small" part of Hilbert space. $\longrightarrow$ QUANTUM COMPRESSION

$$\dim \leq (n+1)^2\, 2^{n(h(p)+\varepsilon')}$$

next time in more detail

⤳ (optimal) rate is <u>von Neumann entropy</u>:

$$S(\rho) = -\mathrm{tr}[\rho \cdot \log \rho] = h(p)$$

PROTOCOL ONLY DEPENDS ON p!!!

<u>Next time</u>: precise language to discuss this & other q. information processing protocols

<u>Symmetries of P_j</u>:

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j \overbrace{V_j \otimes \mathbb{C}^{m(n,j)}}^{P_j \text{ projects here}}$$

$$P_j \cong \bigoplus_{j'} \delta_{jj'} \cdot I$$

For $U \in SU(2)$:

$$U^{\otimes n} \cong \bigoplus_j T_U^{(j)} \otimes I \qquad \text{COMMUTES}$$

⤷ also for $U \in U(2)$, via $U = \underbrace{\sqrt{\det U}}_{\text{scalar}} \cdot \underbrace{\frac{U}{\sqrt{\det U}}}_{\in SU(2)}$

$$U^{\otimes n} \cong \bigoplus_j \boxed{T_U^{(n,j)}} \otimes I$$

representation of $U(2)$: "$V_{n,j}$"

For $\pi \in S_n$?

Schur's lemma

$$R_\pi = \bigoplus_j I \otimes \boxed{R_\pi^{(n,j)}}$$

COMMUTES

representation of S_n: "$W_{n,j}$"

The representations $W_{n,j}$ are irreducible!

Schur-Weyl duality: $\rightarrow$ $\boxed{\text{Pset}}$

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_{n,j} \otimes W_{n,j}$$

irrep for $U(2)$ irrep for S_n

$\hookrightarrow$ $\{P_j\}$ is finest measurement with both symmetries