Intro to Information Theory

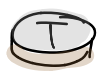
A classical tale:

ALICE — How many bits? → BOB

1 bit

$p = 75\%$

else we make an error w/ 25%



$1-p = 25\%$

n coin flips? Can we do better than $\boxed{1 \, \frac{\text{bit}}{\text{coin flip}}}$?

compression rate

$$\boxed{\#\text{heads} =: k = (p \pm \varepsilon)\, n} \quad \text{(WHP)}$$

$\binom{n}{k}$ possible sequences

$\underbrace{H\ T\ T\ H\ T\ H \cdot H\ H}_{k \text{ heads},\ n-k \text{ tails}}$

asymptotics?

$$x^k (1-x)^{n-k} \binom{n}{k} \leq (x + (1-x))^n = 1$$

$$\xRightarrow{x = \frac{k}{n}} \frac{1}{n} \log \binom{n}{k} \leq -\frac{k}{n} \log \frac{k}{n} - \left(1 - \frac{k}{n}\right) \log \left(1 - \frac{k}{n}\right) = h\left(\frac{k}{n}\right)$$

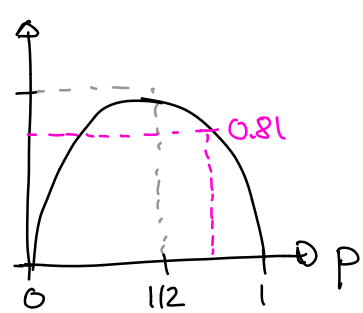
$$\approx h(p)$$

where

$$h(p) = -p \cdot \log p - (1-p) \log (1-p)$$

BINARY ENTROPY

base 2



Compression protocol:

- if $\left|\frac{k}{n} - p\right| > \varepsilon$: FAIL   or send uncompressed
- send k
- send index in list of all coin flip sequences with k heads

$\Rightarrow$ rate: $\cancel{\frac{\log n}{n}} + \boxed{h(p)} + \varepsilon' \quad \frac{\text{bits}}{\text{coin flip}}$ large n

OPTIMAL ASYMPTOTIC RATE

This is what (traditional) IT is all about — also QIT!

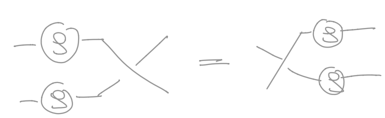
This week: $\boxed{\rho^{\otimes n} \text{ on } (\mathbb{C}^d)^{\otimes n}}$ analog of k?

Today's goal: Spectrum estimation   analog of p

$$\rho^{\otimes n} \xrightarrow{\text{meas.}} \text{eigenvalues } p_1 \geq p_2 \geq \dots \geq p_d \text{ of } \rho$$

Why? (1) Half the way to full tomography.   (2) key difference pure vs mixed ρ

Symmetries: • $[R_\pi, \rho^{\otimes n}] = 0$



but not on Sym^n only

• $U^{\otimes n}\rho^{\otimes n}U^{\dagger\otimes n} = (U\rho U^{\dagger})^{\otimes n} \neq \rho^{\otimes n}$

Same eigenvalues!

↳ Optimal measurement will have both symmetries

WARMUP: $n=2$

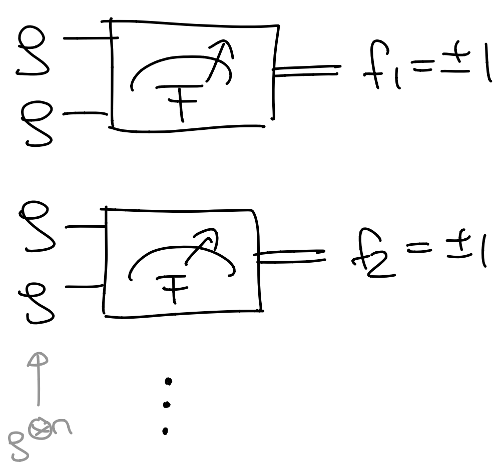
$$(\mathbb{C}^d)^{\otimes 2} = \underbrace{\mathrm{Sym}^2(\mathbb{C}^d)}_{+1} \oplus \underbrace{\Lambda^2(\mathbb{C}^d)}_{-1}$$

Swap operator: $F = R_{1\leftrightarrow 2} = \sum_{a,b} |ab\rangle\langle ba|$

Commutes with $U(d)$ & with $S_2 = \{id, 1\leftrightarrow 2\}$

$$\langle F\rangle = \mathrm{tr}[F\rho^{\otimes 2}] \overset{\text{PSET}}{\underset{\text{SWAP TRICK}}{=}} \mathrm{tr}[\rho^2] = \sum_{i=1}^{d} p_i^2$$

Some information ☺



$$\overline{f_i} \approx \langle F\rangle = \sum_i p_i^2$$

Qubits: $p_1+p_2=1 \rightsquigarrow$ complete sln... but unsatisfactory ↯

breaks symmetry → not good enough for later...

Better solution: Decompose according to $SU(2)$

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j \underbrace{V_j \oplus \cdots \oplus V_j}_{m(n,j) \text{ times}}$$

o.k. ???

$$\cong \bigoplus_j \underbrace{V_j \otimes \mathbb{C}^{m(n,j)}}$$

P_j = projector onto spin-j block

↳ commutes with $U^{\otimes n} \cong \bigoplus_j T_U^{(j)} \otimes I$

also with S_n: $R_\pi = \bigoplus_j I \otimes R_\pi^{(j)}$ ← (ir)reps!

→ Thursday.

Goal: Analyze measurement $\{P_j\}$.

$$\Pr(\text{outcome } j) = \mathrm{tr}[P_j \rho^{\otimes n}]$$

generalizes swap!

w/ $\rho = \begin{pmatrix} p & \\ & 1-p \end{pmatrix}$ $\quad p \geq \frac{1}{2}$

$\rho \mapsto U\rho U^\dagger$

Will see that $\boxed{\frac{1}{2} + \frac{j}{n} \approx p}$

How can we best analyze this measurement?

$\rho^{\otimes n}$ looks like $U^{\otimes n}$!

<u>Fact</u>: Any SU-irrep can be extended to SL-irrep!
det=1

If $\rho > 0$:

$$\tilde{\rho} = \frac{\rho}{\sqrt{\det \rho}} \in SL(2) \rightsquigarrow \rho^{\otimes n} = (\det \rho)^{n/2} \, \tilde{\rho}^{\otimes n}$$

$$\Rightarrow \operatorname{tr}[P_j \rho^{\otimes n}] = (\det \rho)^{n/2} \operatorname{tr}[P_j \tilde{\rho}^{\otimes n}]$$

$$= (\det \rho)^{n/2} \operatorname{tr}\left[T_{\tilde{\rho}}^{(j)} \otimes I_{m(n,j)}\right]$$

$$= (\det \rho)^{n/2} \underbrace{\operatorname{tr}\left[T_{\tilde{\rho}}^{(j)}\right]}_{1} \cdot \underbrace{m(n,j)}_{2}$$

1 $V_j = \mathrm{Sym}^{2j}(\mathbb{C}^2)$: $\boxed{\text{PSET}}$

$$T_{\tilde{\rho}}^{(j)} = \tilde{\rho}^{\otimes 2j}$$

$\|k\rangle\rangle \propto |0\rangle^{\otimes k} |1\rangle^{\otimes (2j-k)}$
+ permutations

$$\Rightarrow \operatorname{tr}\left[T_{\tilde{\rho}}^{(j)}\right] = \sum_{k=0}^{2j} \langle\langle k \| \tilde{\rho}^{\otimes 2j} \| k \rangle\rangle$$

$$= (\det \rho)^{-2j/2} \sum_{k=0}^{2j} p^k (1-p)^{2j-k} \leq (\det \rho)^{-j} \cdot p^{2j} \cdot (2j+1)$$

Plug back in:

$$\operatorname{tr}\left[P_j \rho^{\otimes n}\right] \leq (\det \rho)^{\frac{n}{2}-j} \, p^{2j} \, m(n,j)\,(2j+1)$$

$$= p^{\frac{n}{2}+j} (1-p)^{\frac{n}{2}-j} \, m(n,j) \, (2j+1)$$

② One last tool: Clebsch–Gordan Series

$$V_{j_1} \otimes V_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V_j$$

$U \otimes U$

representation of $SU(2)$

$$\Rightarrow V_j \otimes V_{\frac{1}{2}} = \begin{cases} V_{j+\frac{1}{2}} \oplus V_{j-\frac{1}{2}} & \text{if } j>0 \\ V_{j+\frac{1}{2}} & \text{if } j=0 \end{cases}$$

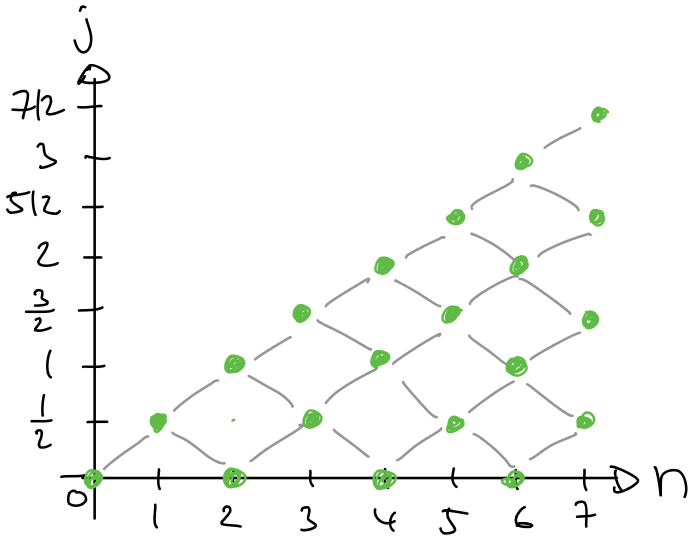
Thus:

$$\mathbb{C}^2 = V_{1/2}$$

$$(\mathbb{C}^2)^{\otimes 2} = V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0$$

$$(\mathbb{C}^2)^{\otimes 3} = (V_1 \oplus V_0) \otimes V_{1/2} = V_{3/2} \oplus \overbrace{V_{1/2} \oplus V_{1/2}}^{m(3,\frac{1}{2})=2}$$

$\vdots$



$m(n,j)$

$= \#$ paths from $(0,0)$ to (n,j)

recursion relation

$$\leq \binom{n}{\frac{n}{2}+j}$$

coin flip

$$\leq 2^{n\,h\left(\frac{1}{2}+\frac{j}{n}\right)}$$

Define $\hat{p} = \frac{1}{2} + \frac{j}{n}$

— WE STOPPED HERE IN CLASS —

Together:

$$\mathrm{tr}\left[P_j \rho^{\otimes n}\right] \leq (2j+1)\, 2^{-n \underbrace{\left[\hat{p}\log\frac{\hat{p}}{p} + (1-\hat{p})\log\frac{1-\hat{p}}{1-p}\right]}_{\delta(\hat{p}\,\|\,p)}}$$

BINARY RELATIVE ENTROPY

Pinsker's inequality: $\delta(\hat{p}\,\|\,p) \geq \frac{2}{\ln 2}(\hat{p}-p)^2$

Result:

$$\boxed{\operatorname{tr}\left[P_j \rho^{\otimes n}\right] \leq (2j+1)\, 2^{-n \frac{2}{\ln 2}(\hat{p}-p)^2}}$$

Keyl & Werner

outcome $j \longrightarrow$ estimate $\hat{p} = \frac{1}{2} + \frac{j}{n}$

- $\Pr\left(|\hat{p}-p| > \varepsilon\right) \leq \sum_{j : \left|\frac{1}{2}+\frac{j}{n}-p\right| > \varepsilon} \operatorname{tr}\left[P_j \rho^{\otimes n}\right]$

$$\leq n(2j+1)\, 2^{-n \frac{2}{\ln 2} \varepsilon^2}$$ exponentially small

- $\operatorname{rank} P_j \sim 2^{n h(\hat{p})} \sim 2^{n(h(p) \pm \varepsilon')}$ $\to$ compression

$h(p) = -\operatorname{tr} \rho \log \rho = S(\rho)$

von Neumann entropy of ρ