Measurements & Estimation

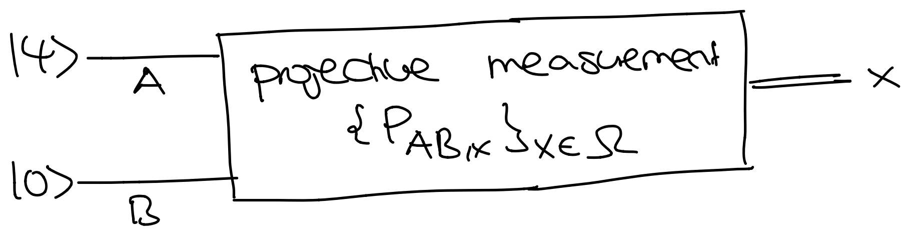
Observable $X = \sum_{x \in \Omega} x \cdot P_x \longleftrightarrow \{P_x\}_{x\in\Omega}$ projective measurement

eigenvalues

projectors onto eigenspaces

Born's rule: $\Pr(\text{outcome } x) = \langle \psi | P_x | \psi \rangle$

More general measurements are possible!



(Equ.: couple to measurement device B, apply interacting time evolution $U_{AB}(t)$, read off result at apparatus)

$$\Pr(\text{outcome } x) = \left(\langle \psi|_A \otimes \langle 0|_B\right) P_{AB,x} \left(|\psi\rangle_A \otimes |0\rangle_B\right)$$

$$= \langle \psi_A| \underbrace{\left[(I_A \otimes \langle 0|_B) P_{AB,x} (I_A \otimes |0\rangle_B \right]}_{=: Q_x} |\psi_A\rangle$$

$$= \langle \psi_A | Q_x | \psi_A \rangle$$

Ex: $P_{AB} = \left(\frac{|00\rangle \pm |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| \pm \langle 11|}{\sqrt{2}}\right)$

$\Rightarrow (I_A \otimes \langle 0|_B)\, P_{AB}\, (I_A \otimes |0\rangle_B)$

$= \frac{1}{2}|0\rangle\langle 0|$

Same one!

$$\boxed{\begin{array}{l} \bullet\ Q_x \geq 0 \\ \bullet\ \sum_x Q_x = I_A \end{array}} = (I_A \otimes \langle 0|_B) \underbrace{\sum_x P_{AB,x}}_{= I_{AB}} (I_A \otimes |0\rangle_B) \checkmark$$

POVM measurement

Any such $\{Q_x\}$ is physical! → pset

Truely more general!

E.g., $\{\frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|+\rangle\langle +|, \frac{1}{2}|-\rangle\langle -|\}$.

not orthogonal

What if Ω is infinite (e.g. continuous)?

Continuous POVM: Ω space of outcomes, dx measure

$$\boxed{\begin{array}{l} \bullet\ Q_x \geq 0 \\ \bullet\ \int dx\, Q_x = I \end{array}}$$

More subtle if $\mathcal{H}$ is infinite-dim.
→ "pos. op.-valued measures" (POVMs!!)
e.g. X-meas. in QM !!!

Born's rule:

$\langle\psi|Q_x|\psi\rangle$ = prob. density of outcome $x \in \Omega$

ie.

$$\Pr(\text{outcome } x \in S) = \int_S dx \, \langle\psi|Q_x|\psi\rangle$$

$$E[f(x)] = \int dx \, \langle\psi|Q_x|\psi\rangle \, f(x)$$

Today's goal: Estimating a pure state

$$|\psi\rangle \xrightarrow[???]{\text{meas.}} \text{"}\psi\text{"}$$

q. state — classical description

- $|\psi\rangle \longrightarrow \text{"}\psi\text{"} \longrightarrow |\psi\rangle^{\otimes 2}$ ↯ cloning
- $(\langle\psi|^{\otimes n})(|\phi\rangle^{\otimes n}) = \langle\psi|\phi\rangle^n \longrightarrow 0$ if $\psi \neq \phi$

many copies are almost orthogonal! ⊛

Goal: Design continuous POVM $\{Q_{\hat\psi}\}$ s.th.

$$|\psi\rangle^{\otimes n} \longrightarrow \boxed{\text{POVM}} \Longrightarrow \hat\psi \approx \psi$$

$\uparrow$ estimate

⊛ suggests: $\left\{ \frac{1}{Z} |\hat\psi\rangle\langle\hat\psi|^{\otimes n} \right\}$ is a good POVM!

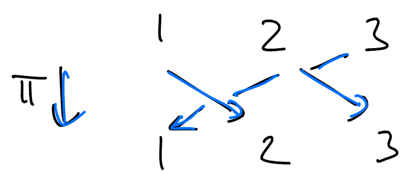
$$|\psi\rangle \in \mathbb{C}^d \longrightarrow \boxed{|\psi\rangle^{\otimes n}} \in (\mathbb{C}^d)^{\otimes n}$$

invariant under permuting subsystems

???

Symmetric $\boxed{\text{group}}$ S_n:

- elements are permutations $\pi : \{1,\dots,n\} \longrightarrow \{1,\dots,n\}$
- $\# = n!$



For $\pi \in S_n$, define operator on $(\mathbb{C}^d)^{\otimes n}$:

$$R_\pi |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle = |\psi_{\pi^{-1}(1)}\rangle \otimes \cdots \otimes |\psi_{\pi^{-1}(n)}\rangle$$

$$\Longrightarrow \boxed{R_{id} = I \quad \& \quad R_\pi R_\tau = R_{\pi\tau}}$$

???

$\boxed{\text{representation}}$ of S_n on $(\mathbb{C}^d)^{\otimes n}$

Symmetric subspace: Bosonic Fock space w/ n particles

$$\mathrm{Sym}^n(\mathbb{C}^d) = \{ |\Phi\rangle \in (\mathbb{C}^d)^{\otimes n},\ R_\pi |\Phi\rangle = |\Phi\rangle \ (\forall \pi) \}$$

$\in$

$|\psi\rangle^{\otimes n}$

Ex: $\mathrm{Sym}^2(\mathbb{C}^2)$ spanned by

$$|00\rangle, \quad |11\rangle, \quad \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

not of this form!

Missing: $|10\rangle - |01\rangle / \sqrt{2}$.

Symmetrize:

$$\Pi_n = \frac{1}{n!} \sum_{\pi \in S_n} R_\pi$$

projector onto symmetric subspace

- $|\Phi\rangle$ arbitrary $\Rightarrow \Pi_n |\Phi\rangle \in \mathrm{Sym}^n(\mathbb{C}^d)$
- $|\Phi\rangle \in \mathrm{Sym}^n(\mathbb{C}^d) \Rightarrow \Pi_n |\Phi\rangle = |\Phi\rangle$

e.g. $R_\pi(\Pi_n |\Phi\rangle) = \frac{1}{n!} \sum_{\tau \in S_n} R_{\pi\tau} |\Phi\rangle = \frac{1}{n} \sum_{\tilde{\pi} \in S_n} R_{\tilde{\pi}} |\Phi\rangle$

$= \Pi_n |\Phi\rangle$. ✓

Basis of $\mathrm{Sym}^n(\mathbb{C}^d)$:

$$\Pi_n |\underbrace{1 \cdots 1}_{n_1 \text{ many}}, \underbrace{2 \cdots 2}, \underbrace{3 \cdots 3}, \underbrace{d \cdots d}_{n_d \text{ many}}\rangle \qquad \sum_{i=1}^{d} n_i = n$$

$n = (n_1, \ldots, n_d)$ occupation numbers

$$\Rightarrow \dim \mathrm{Sym}^n(\mathbb{C}^d) = \binom{n+d-1}{n} = \frac{(n+d-1)!}{n!\,(d-1)!}$$

$$= \mathrm{tr}\, \Pi_n$$

Recall: • $\langle \psi|^{\otimes n} |\phi\rangle^{\otimes n} \neq 0$ in general

yet: • not all $|\Phi\rangle \in \mathrm{Sym}^n$ of form $|\psi\rangle^{\otimes n}$

Claim:

$$\boxed{\Pi_n = \binom{n+d-1}{n} \int d\psi \; |\psi\rangle^{\otimes n} \langle\psi|^{\otimes n}}$$

(unique) probability measure s.th. invariant under $|\psi\rangle \mapsto U|\psi\rangle$ $\forall U \in U(d)$ unitary

"Haar measure"

i.e. $\int d\psi \; f(|\psi\rangle) = \int d\psi \; f(U|\psi\rangle)$

i.e. rotationally invariant on unit sphere S^{2n-1}

proof likely on Tuesday

- $|\psi\rangle^{\otimes n}$ is "overcomplete basis": $|\Phi\rangle \in \mathrm{Sym}^n$

$$\Longrightarrow |\Phi\rangle = \Pi_n |\Phi\rangle = (\ldots) \underbrace{\int d\psi \; |\psi\rangle^{\otimes n} \cdot \underbrace{(\langle\psi|^{\otimes n} |\Phi\rangle)}_{\text{scalar}}}_{\text{linear combination}}$$

- $Q_{\hat\psi} = \binom{n+d-1}{n} |\hat\psi\rangle^{\otimes n} \langle\hat\psi|^{\otimes n}$ is a POVM on Sym^n

$$\int d\hat\psi \; Q_{\hat\psi} = \Pi_n$$

How to quantify performance of this estimator?

$$\boxed{|\langle\hat\psi|\psi\rangle|^{2k}} \qquad \text{overlap}$$

Average performance:

$$\mathbb{E}\,|\langle\hat{\psi}|\psi\rangle|^{2k}$$

Born

$$= \int d\hat{\psi}\ \langle\psi|^{\otimes n} Q_{\hat{\psi}} |\psi\rangle^{\otimes n}\ |\langle\hat{\psi}|\psi\rangle|^{2k}$$

$$= \binom{n+d-1}{n} \int d\hat{\psi}\ |\langle\hat{\psi}|\psi\rangle|^{2(n+k)}$$

$$= \binom{n+d-1}{n} \langle\psi|^{\otimes(n+k)} \left[\int d\hat{\psi}\ |\hat{\psi}\rangle^{\otimes(n+k)}\langle\hat{\psi}|^{\otimes(n+k)}\right] |\psi\rangle^{\otimes(n+k)}$$

$$= \binom{n+d-1}{n}\binom{n+k+d-1}{n+k}^{-1} \underbrace{\langle\psi|^{\otimes(n+k)}\, \Pi_{n+k}\, |\psi\rangle^{\otimes(n+k)}}_{=1}$$

$$= \frac{(n+d-1)!}{n!\,\cancel{(d-1)!}}\ \frac{(n+k)!\,\cancel{(d-1)!}}{(n+k+d-1)!}$$

$$= \frac{(n+d-1)\cdots(n+1)}{(n+k+d-1)\cdots(n+k+1)} \geq \left[\frac{n+1}{n+k+1}\right]^{d-1}$$

$$= \left[1 - \frac{k}{n+k+1}\right]^{d-1} \geq 1 - \frac{(d-1)k}{n+k+1}$$

$$\geq 1 - \frac{dk}{n}.$$

Pset: Means that $|\psi\rangle$, $|\hat{\psi}\rangle$ almost indist. by any measurement!

SUCCESS! High overlap if $n \gg d \cdot k$.