Quantum circuits, swap test, quantum Schur transform

Lecture 10 *Michael Walter, Stanford University*

These lecture notes are not proof-read and are offered for your convenience only. They include additional detail and references to supplementary reading material. I would be grateful if you email me about any mistakes and typos that you find.

In the past two weeks, we used an important tool, the decomposition

$$(\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_j \otimes \mathbb{C}^{m(n,j)} \tag{10.1}$$

of the n-qubit Hilbert space into irreducible representations of SU(2). We used the "Schur-Weyl toolbox" obtained in this way to solve the spectrum estimation problem, various data compression problems, and to study entanglement transformations (lectures 5, 6, 8 and 9). A fundamental role was played by the the projections P_j onto the different sectors. But how would we realize these projections in practice?

Recall that the notation $\cong$ in eq. (10.1) refers to a unitary intertwiner

$$(\mathbb{C}^2)^{\otimes n} \to \bigoplus_j V_j \otimes \mathbb{C}^{m(n,j)}.$$

The n-qubit Hilbert space on the left-hand side has the product basis

$$|x_1, \dots, x_n\rangle = |x_1\rangle \otimes \dots \otimes |x_n\rangle,$$

while the right-hand side has a natural "Schur-Weyl basis" labeled by

$$|j, m, k\rangle$$

where $j \in \{\dots, \frac{n}{2} - 1, \frac{n}{2}\}$, $m \in \{-j, \dots, j\}$, $k \in \{1, \dots, m(n,j)\}$. Since the values of m and k are constrained by j, the right-hand side space is *not* a tensor product. However, we can safely think of it as a *subspace* of the tensor product space

$$\mathbb{C}^n \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{2^n},$$

since (i) there are at most n options for j, (ii) the dimension of V_j is $\frac{2}{j} + 1 \leq n + 1$, and (iii) certainly $m(n,j) \leq 2^n$. Thus, we obtain an isometry

$$U_{\text{Schur}} : (\mathbb{C}^2)^{\otimes n} \longrightarrow \mathbb{C}^n \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{2^n} \tag{10.2}$$

This transformation is called the *quantum Schur transform* (fig. 16, (a)).

Why is this convenient? The isometry nicely separates the three pieces of information that we care about – the spin j and the corresponding vectors in V_j and in $\mathbb{C}^{m(n,j)}$ – into different subsystems. For example, we can now implement the spin measurement $\{P_j\}$ by first applying U_{Schur} and then measuring the first subsystem. In other words,

$$P_j = U_{\text{Schur}}^\dagger \left(|j\rangle\langle j| \otimes \mathbb{1} \otimes \mathbb{1}\right) U_{\text{Schur}}.$$

This is visualized in fig. 16, (b). The goal of today's lecture will be to design a *quantum circuit* for the quantum Schur transform.

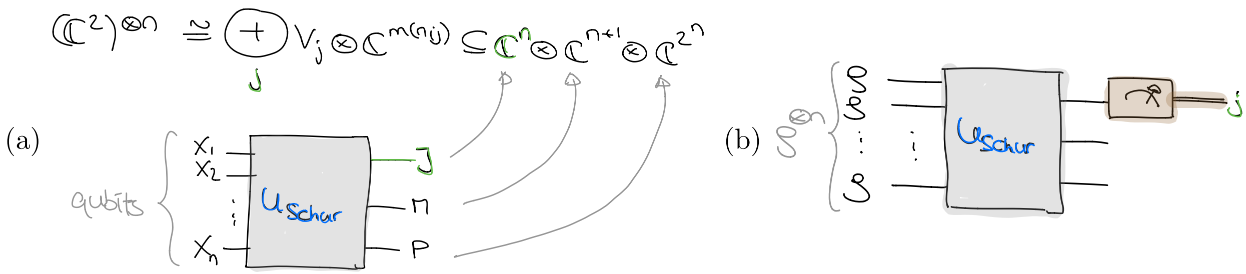


Figure 16: (a) The Schur transform (10.2). (b) We can implement the measurement $\{P_j\}$ by first applying the Schur transform and then measuring the j-system.

10.1 Quantum circuits

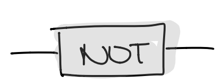
Just like we typically describe computer programs or algorithms in terms of simple elementary instructions, we are interested in constructing a unitary transformation U of interest from "simple" building blocks. These building blocks are *quantum gates*, i.e., unitary operations that involve only a smaller number of qubits (or qudits). We obtain a *quantum circuit* by connecting the output of some quantum gates by "wires" with the inputs of others. We will also allow *measurements* of individual qubits in the standard basis $\{|i\rangle\}$ as well as the *initialization* of qubits in basis states $|i\rangle$. For example, the circuit in fig. 17 first adds a qubit in state $|0\rangle$, then performs the unitary

$$(U_3 \otimes U_4)\,(\mathbb{1}_{\mathbb{C}^2} \otimes U_2 \otimes \mathbb{1}_{\mathbb{C}^2})\,(U_1 \otimes \mathbb{1}_{\mathbb{C}^2} \otimes \mathbb{1}_{\mathbb{C}^2})$$

and then measures one of the qubits. In the absence of measurements and initializations, a quantum circuit performs a unitary transformation from the input qubits to the output qubits. In the absence of measurements alone, the quantum circuit implements an isometry from the input qubits to the outputs qubits.

Remark. *The number of gates in a quantum circuit is known as the* (gate) complexity *of that circuit. Intuitively, the higher the complexity the longer it would take a quantum computer to run this circuit. This is because we expect that a quantum computer, in completely analogy to a classical computer, will be able to implement each gate and measurement in a small, fixed amount of time. Much of the field of* quantum computation *is concerned with finding quantum circuits and algorithms of minimal complexity – with a particular emphasis on finding quantum algorithms that outperform all known classical algorithms. For example, Peter Shor's famous factoring algorithm outperforms all known classical factoring algorithms. Just like quantum information theory, this is a very rich subject. In this course, we only have time for a glance, but I encourage you to look at Nielsen and Chuang (2002), Kitaev et al. (2002) for further detail if you are interested in this subject.*

To practice, let us consider some interesting gates. For any single-qubit unitary U, there is a corresponding *single-qubit gate*. For example, the Pauli X-operator $X = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ gives rise to the so-called X-gate or *NOT-gate*



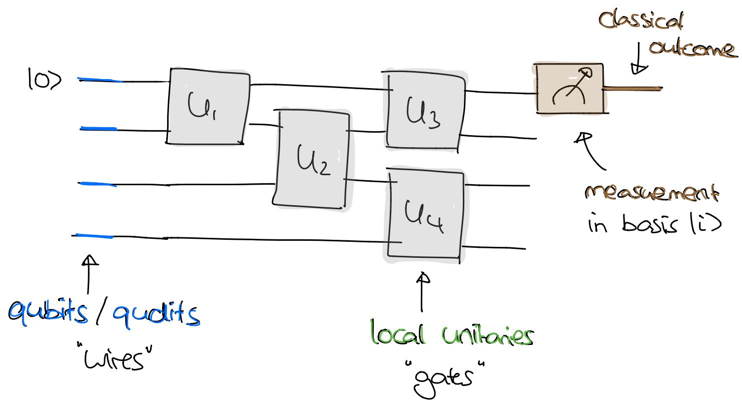
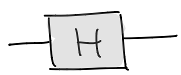


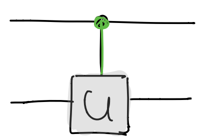
Figure 17: Illustration of a quantum circuit, composed of four unitary quantum gates and a single measurement. The first qubit is initialized in state $|0\rangle$ and the other three wires are inputs to the circuit.

which maps $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$. Another example is the so-called *Hadamard gate*



which maps $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$. Written as a unitary matrix, $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.

Single-qubit gates are not enough – for example, they do not allow us to create an entangled state starting from product states. A powerful class of gates can be obtained by performing a unitary transformation U depending on the value of a *control qubit*. This is a standard but slightly misleading figure of speech, since we do not actually want to measure the value of the control qubit. To be more precise, we define the *controlled unitary gate*



by

$$\begin{aligned} \mathrm{C}U(|0\rangle \otimes |\psi\rangle) &= |0\rangle \otimes |\psi\rangle\,, \\ \mathrm{C}U(|1\rangle \otimes |\psi\rangle) &= |0\rangle \otimes (U|\psi\rangle) \end{aligned} \tag{10.3}$$

(and extend by linearity). It is easy to see that $\mathrm{C}U$ is indeed a unitary (indeed, $\mathrm{C}(U^\dagger)$ is its inverse).

Remark 10.1. *More generally, if U_0, U_1 are two unitaries then we can define a controlled unitary by $|x\rangle \mapsto U_x|x\rangle$. We will use this below when constructing a quantum circuit for the Clebsch-Gordan transformation.*

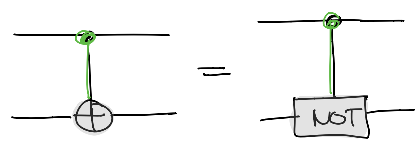
For example, if U is the NOT-gate then the *controlled not (CNOT) gate* maps

$$\begin{aligned}
\mathrm{CNOT}\,|0,0\rangle &= |0,0\rangle\,, \\
\mathrm{CNOT}\,|0,1\rangle &= |0,1\rangle\,, \\
\mathrm{CNOT}\,|1,0\rangle &= |1,1\rangle\,, \\
\mathrm{CNOT}\,|1,1\rangle &= |1,0\rangle\,,
\end{aligned}$$

i.e.,

$$\mathrm{CNOT}\,|x,y\rangle = |x, x \oplus y\rangle\,,$$

where, as usual, $\oplus$ denotes addition modulo 2. This explains why the CNOT gate is often denoted by

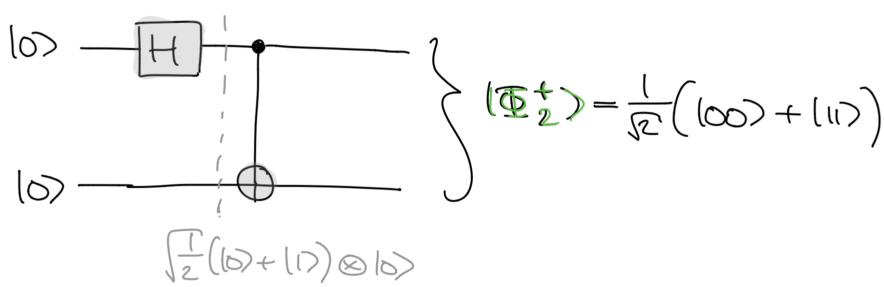


Using these ingredients, we can already build a number of interesting circuits.

Remark. *In fact, any N-qubit unitary can be to arbitrarily high fidelity approximated by quantum circuits composed only of CNOT-gates and single qubit gates. We say, that the CNOT gate together with the single qubit gates form a* universal gate set*. (In fact, CNOT together with a finite number of single qubit gates suffices.)*

Entanglement and teleportation

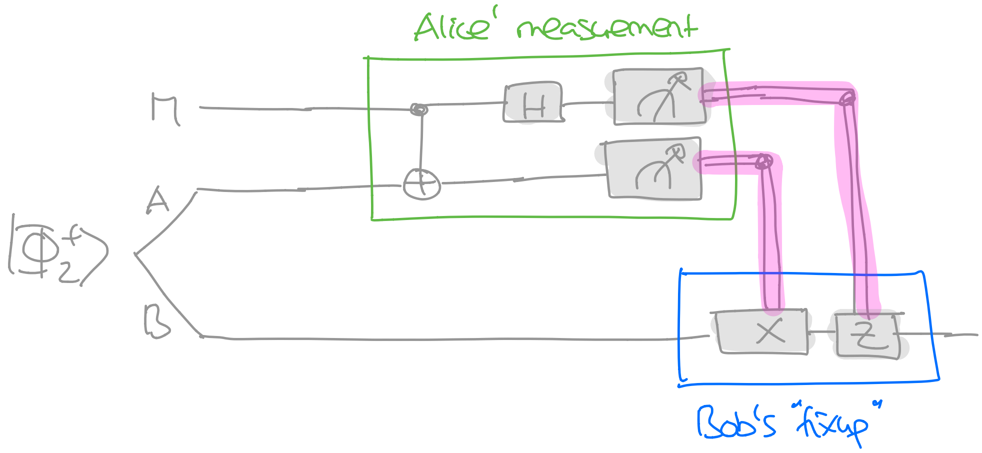
For example, consider the following circuit:



It is plain that this creates an ebit starting from the product state $|00\rangle$. More generally, for each product basis state $|xy\rangle$ the circuit produces one of the four maximally entangled basis vectors $|\phi_k\rangle$ from eq. (9.2) that we used in teleportation. Indeed, the circuit maps

$$|x,y\rangle \mapsto \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^x\,|1\rangle\right) \otimes |y\rangle = \frac{1}{\sqrt{2}}\left(|0,y\rangle + (-1)^x\,|1,y\rangle\right).$$

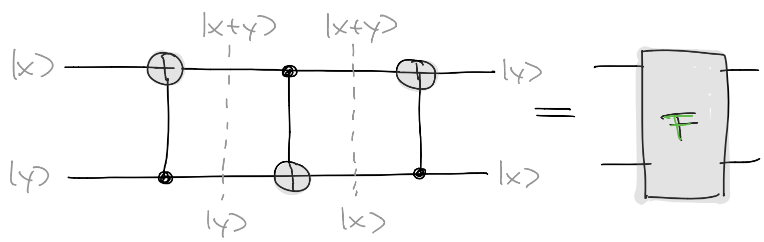
As a consequence, this allows us to write down a more detailed version of the teleportation circuit from last time (fig. 13):



The doubled wires (pink) denote the classical measurement outcomes (two bits x and y, corresponding to the single integer $k \in \{0,1,2,3\}$ from last time). It is a fun exercise to verify that this circuit works as desired, i.e., that it implements an identity map from the input qubit M to the output qubit B.

10.2 The swap test

We can implement the swap unitary $F: |xy\rangle \mapsto |yx\rangle$ by a quantum circuit composed of three CNOTs.



This is called the *swap gate*.

We can also write down a corresponding *controlled swap gate*, defined as in eq. (10.3) for $U = F$. Note that this is a *three* qubit gate. In problem 4.5, you will find a quantum circuit for the controlled swap gate that involves only single-qubit and two-qubit gates.

When we started studying the spectrum estimation problem in lecture 5, we first considered the case that we were given $n = 2$ two copies of our state as a "warmup" in example 5.3. The idea was that the two-qubit Hilbert space decomposes into the symmetric (triplet) and antisymmetric (singlet) subspaces,

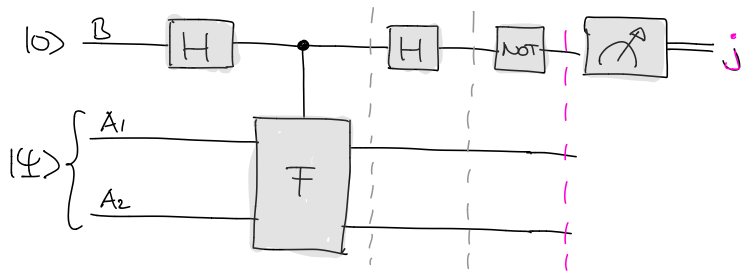
$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathrm{Sym}^2(\mathbb{C}^2) \oplus \bigwedge^2(\mathbb{C}^2),$$

which is of course a special case of eq. (10.1) since the triplet is a spin-1 irrep and the singlet a spin-0 irrep of SU(2). The swap operator F acts by +1 on the triplet but by by −1 on the singlet, i.e.,

$$F = P_1 - P_0,$$

so measuring F is completely equivalent to performing the projective measurement $\{P_0, P_1\}$.

How can we implement this measurement by a quantum circuit? Consider the following circuit, which uses the *controlled* swap gate discussed above:



(10.4)

Why does this circuit perform the desired measurement? Suppose that we initialize the B-wire in state $|0\rangle$ and the A-qubits in some arbitrary state $|\Psi\rangle$. The Hadamard gate sends $|0\rangle \mapsto |+\rangle$ and so the quantum state right after the controlled swap gate (first dashed line) is equal to

$$\frac{1}{\sqrt{2}} \left(|0\rangle_B \otimes |\Psi\rangle_A + |1\rangle_B \otimes F\, |\Psi\rangle_A \right)$$

After the second Hadamard gate (second dashed line), we obtain

$$\begin{aligned}
&\frac{1}{2} \left[(|0\rangle_B + |1\rangle_B) \otimes |\Psi\rangle_A + (|0\rangle_B - |1\rangle_B) \otimes F\, |\Psi\rangle_A \right] \\
&= |0\rangle_B \otimes \frac{\mathbb{1} + F}{2} |\Psi\rangle_A + |1\rangle_B \otimes \frac{\mathbb{1} - F}{2} |\Psi\rangle_A \\
&= |0\rangle_B \otimes \Pi_2\, |\Psi\rangle_A + |1\rangle_B \otimes (\mathbb{1} - \Pi_2)\, |\Psi\rangle_A \\
&= |0\rangle_B \otimes P_1\, |\Psi\rangle_A + |1\rangle_B \otimes P_0\, |\Psi\rangle_A\,,
\end{aligned}$$

where Π_2 is the projector onto symmetric subspace, which for $n = 2$ qubits is nothing but the spin-1 projection P_1. The last NOT simply relabels $|0\rangle_B \leftrightarrow |1\rangle_B$, leading to

$$|1\rangle_B \otimes P_1\, |\Psi\rangle_A + |0\rangle_B \otimes P_0\, |\Psi\rangle_A\,.$$

In summary, the quantum circuit achieves the following task: It transforms an arbitrary input state $|\Psi\rangle_A$ into the following state right before the measurement of the B-qubit (last, pink dashed line)

$$|\Psi\rangle_A \mapsto \sum_{j=0,1} |j\rangle_B \otimes P_j\, |\Psi\rangle_A\,.$$

Hence

$$\Pr(\text{outcome } j) = \langle \Psi_A | P_j | \Psi_A \rangle\,,$$

and the post-measurement state on the A-qubits is proportional to $P_j\, |\Psi\rangle_A$. Thus, we have successfully implemented the measurement $\{P_0, P_1\}$. The quantum circuit (10.4) is known as the *swap test*.

Applications

The swap test has many applications:

- If we choose $\rho^{\otimes 2}$ as input state for the A-qubits, then

$$\Pr(\text{outcome } j) = \operatorname{tr}\left[P_j \rho^{\otimes 2}\right],$$

i.e.,

$$\Pr(\text{outcome } 1) = \frac{1}{2}\left(1 + \operatorname{tr}\rho^2\right) = 1 - \Pr(\text{outcome } 0),$$

from which we can learn information about the spectrum of ρ. In particular, it allows us to estimate the *purity* $\operatorname{tr}\rho^2$ of the unknown quantum state (cf. example 5.3).

This was our original motivation for implementing the swap test.

- If we choose $|\psi\rangle_{A_1} \otimes |\phi\rangle_{A_2}$ as input state, then

$$\begin{aligned}\Pr(\text{outcome } 1) &= \frac{1}{2}\left(1 + \langle \psi_{A_1} \otimes \phi_{A_2} | F | \psi_{A_1} \otimes \phi_{A_2} \rangle\right) \\ &= \frac{1}{2}\left(1 + \langle \psi_{A_1} \otimes \phi_{A_2} | \phi_{A_1} \otimes \psi_{A_2} \rangle\right) = \frac{1}{2}\left(1 + |\langle \psi | \phi \rangle|^2\right),\end{aligned} \tag{10.5}$$

which allows us to estimate the overlap $|\langle\psi|\phi\rangle_{A_1}|$ between the pure states $|\psi\rangle$ and $|\phi\rangle$. Thus, the swap test can be used to test two unknown pure states for equality.

The swap test can be readily generalized to qu*d*its.

Remark. *There is a fun application of the swap test known as* quantum fingerprinting*, which we might discuss in class if there is enough time (Buhrman et al., 2001): The rough idea goes as follows: We can find 2^n many pure states $|\psi(\vec{x})\rangle \in \mathbb{C}^{cn}$, indexed by classical bit strings $\vec{x}$ of length n, with pairwise overlaps*

$$\langle \psi(\vec{x}) | \psi(\vec{y}) \rangle \leq \frac{1}{2}.$$

Here $c > 0$ is some constant. Thus the quantum states live in a space of only order $\log n$ *many qubits! (How can we justify the existence of such vectors? One way is to just choose them at random and estimate probabilities using a more refined version of our calculations for the symmetric subspace, see Harrow (2013) for more detail.) If we perform k swap tests on $|\psi(\vec{x})\rangle^{\otimes k} \otimes |\psi(\vec{y})\rangle^{\otimes k}$ then we obtain*

$$\vec{x} \neq \vec{y} \quad \Rightarrow \quad \Pr(\textit{outcome 1 for all } k \textit{ swap tests}) = \left(\frac{3}{4}\right)^k \approx 0$$

Thus the probability of outcome 1 is arbitrarily small, controlled only by the parameter k (but not n). In this sense, we can use the states $|\psi(\vec{x})\rangle$ as short "fingerprints" for the classical bit strings $\vec{x}$. The latter are require n bits to specify, while the fingerprints only need order $k \log n$ many qubits (this is not even optimal, but sufficient for our purposes).

Remarkably, while this allows us to test the fingerprints pairwise for equality with high certainty, it is not *possible to determine the original bitstring $|\vec{x}\rangle$ from its fingerprint $|\psi(\vec{x})\rangle$ to good fidelity. This is ensured by the same Holevo bound mentioned last time in section 9.3, which ensures that we cannot communicate more than one classical bit by sending over a single qubit (in the absence of ebits).*

10.3 The quantum Schur transform

Now that we have acquired some familiarity with quantum circuitry, we will turn towards solving our actual goal for today – finding a quantum circuit for the Schur transform (10.2),

$$U_{\text{Schur}} \colon (\mathbb{C}^2)^{\otimes n} \cong \bigoplus_j V_j \otimes \mathbb{C}^{m(n,j)} \longrightarrow \mathbb{C}^n \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{2^n}$$

(cf. fig. 16). We'll follow the exposition in Christandl (2010).

The Clebsch-Gordan isometry

In lecture 6, we obtained the multiplicities $m(n,j)$ by successively applying the Clebsch-Gordan rule,

$$V_j \otimes V_{1/2} \cong \bigoplus_{j'=j-\frac{1}{2}}^{j+\frac{1}{2}} V_{j'}. \tag{10.6}$$

From your quantum mechanics class you know that the spin-j representation V_j has a basis $|j,m\rangle$ with $m = -j,\dots,j$. The matrix elements of the basis transformation corresponding to (10.6) are known as the Clebsch-Gordan coefficients. They can packaged up in terms of unitary 2×2-matrices $U(j,m)$ such that

$$|j,m\rangle \otimes |\frac{1}{2}, s\rangle = \sum_{s'=-\frac{1}{2}}^{\frac{1}{2}} U(j,m)_{s,s'} \, |j+s', m+s\rangle\,. \tag{10.7}$$

for $s = \pm\frac{1}{2}$.

Remark. *Why is this the case, and how can these coefficients be computed? The defining property of the basis vectors $|j,m\rangle$ of V_j is that*

$$\widetilde{Z}\,|j,m\rangle = 2m\,|j,m\rangle\,, \tag{10.8}$$

where $\widetilde{Z}$ denotes the action of the "generator" Z of $\mathrm{SU}(2)$, as discussed in remark 5.4. On the other hand, if we consider the action of the generator on the tensor product $V_j \otimes V_{1/2}$, then the generator Z acts by

$$\left(\widetilde{Z} \otimes \mathbb{1} + \mathbb{1} \otimes \widetilde{Z}\right)\left(|j,m\rangle \otimes |\frac{1}{2}, s\rangle\right) = 2(m+s)\left(|j,m\rangle \otimes |\frac{1}{2}, s\rangle\right).$$

By comparing with eq. (10.8), this means that $|j,m\rangle \otimes |\frac{1}{2}, s\rangle$ can indeed be written as a linear combination of $|j',m'\rangle$ with $m' = m+s$ – that is, in the form of eq. (10.7).

How can the coefficients be determined? First, note that the only way of obtaining $m' = j + \frac{1}{2}$ is by choosing $m = j$ and $s = \frac{1}{2}$. Thus,

$$|j+\frac{1}{2}, j+\frac{1}{2}\rangle = |j,j\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle\,. \tag{10.9}$$

Now you will remember from your quantum mechanics that the spin lowering operator $S_\pm = X - iY$ acts by

$$\widetilde{S}_-\,|j,m\rangle = 2\sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle\,.$$

By successively acting with S_- on eq. (10.9) (i.e., by $\widetilde{S}_-$ on the left and by $\widetilde{S}_- \otimes \mathbb{1} + \mathbb{1} \otimes \widetilde{S}_-$ on the right), this allows us to obtain an expression of the form

$$|j + \frac{1}{2}, m'\rangle = \# \, |j, m' - \frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle + \# \, |j, m' + \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle$$

for some coefficients #. Thus we have identified $V_{j+\frac{1}{2}}$ in $V_j \otimes V_{1/2}$. Next, we observe that

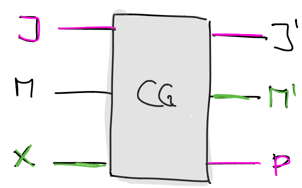
$$|j - \frac{1}{2}, j - \frac{1}{2}\rangle = \# \, |j, j-1\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle + \# \, |j, j\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle \tag{10.10}$$

is now uniquely determined by orthogonality to $|j + \frac{1}{2}, j - \frac{1}{2}\rangle$. We can now similarly obtain the coefficients in

$$|j - \frac{1}{2}, m'\rangle = \# \, |j, m' - \frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle + \# \, |j, m' + \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle$$

by successfully applying the action of the generator S_- to eq. (10.10).

We now define the Clebsch-Gordan isometry U_{CG},



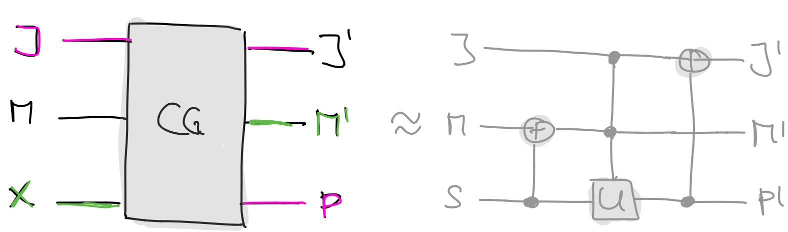
as the isometry that sends

$$|j, m, x\rangle \mapsto |j, m\rangle \otimes |\frac{1}{2}, s\rangle \mapsto U(j,m)_{s,\frac{1}{2}} \, |j + \frac{1}{2}, m + s\rangle \otimes |+\rangle$$
$$+ U(j,m)_{s,-\frac{1}{2}} \, |j - \frac{1}{2}, m + s\rangle \otimes |-\rangle \, ,$$

where we first relabel the standard basis $|x\rangle$ of $\mathbb{C}^2$ to $|\frac{1}{2}, s\rangle$ of $V_{1/2}$, with $s := \frac{1}{2} - x \in \{\pm\frac{1}{2}\}$, and then apply the Clebssch-Gordan transformation. (To be precise, we should restrict the possible values of j to some $j_{\max}$ to obtain a finite matrix.)

What is the meaning of the output p? In eq. (10.7), the left-hand side spin j was fixed, but the spin j is now part of the input. Since the same j' can be obtained from two possible values of j, we use an additional output p to remember the "direction" by which we arrived at j' (that is, $j' = j + \frac{p}{2}$). Only then is U_{CG} an isometry.

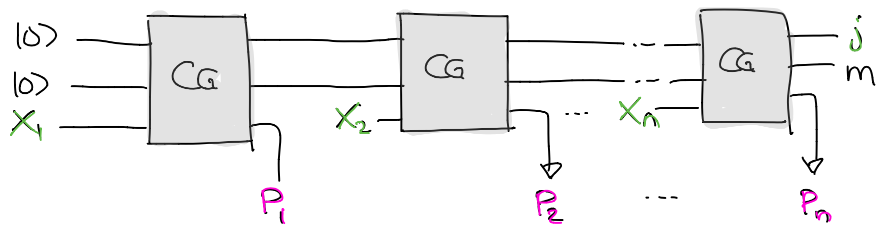
Schematically, the Clebsch-Gordan isometry U_{CG} can be implemented by a quantum circuit of the following form



where the middle part uses the slightly more general notion of a controlled unitary described in remark 10.1, mapping $|j, m, s\rangle$ to $|j, m\rangle \otimes U(j,m) \, |s\rangle$.

The quantum Schur transform

We now obtain the quantum Schur transform U_{Schur} from eq. (10.2) by composing n Clebsch-Gordan transformations:



We input the n qubits into the wires $X_1, \ldots, X_n$ and the output consists of J, M, and $P = (P_1, \ldots, P_n)$. A moments thought shows that this indeed implements the desired transformation.

In particular, we can implement the spectrum estimation measurement $\{P_j\}$ by first applying the quantum Schur transform and then measuring the J-system in the standard basis (as in fig. 16, (b)).

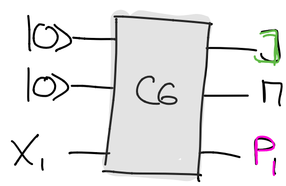
Remark. *We can expand*

$$U_{Schur} |\Psi\rangle = \sum_j \psi_{j,m,\vec{p}} |j\rangle_J \otimes |m\rangle_M \otimes |\vec{p}\rangle_P \,,$$

where $\vec{p} \in \{\pm\}^n$. *Then* $\psi_{j,m,\vec{p}} \neq 0$ *only if* $\vec{p}$ *is a sequence* $|+ - + + - \ldots\rangle_P$ *that corresponds to a path from* $(0,0)$ *to* (n,j) *in fig. 9.*

At last, let us discuss some concrete examples to make sure that we fully understand what is going on:

Example (n=1)**.** *For a single qubit, the Schur transform is completely trivial:*

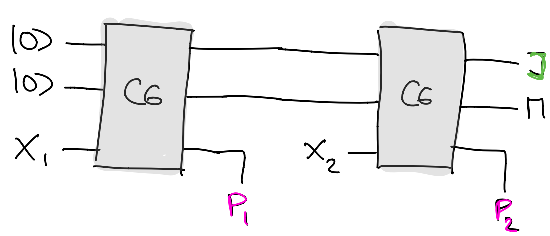


It maps

$$|0\rangle_X \mapsto |\frac{1}{2}\rangle_J \otimes |\frac{1}{2}\rangle_M \otimes |+\rangle_P$$
$$|1\rangle_X \mapsto |\frac{1}{2}\rangle_J \otimes |-\frac{1}{2}\rangle_M \otimes |+\rangle_P$$

Note that the P-system is always in the $|+\rangle$ *state, corresponding to the path* $(0,0) \to (\frac{1}{2}, 1)$.

Example (n=2)**.** *For two qubits, the Schur transform*



maps

$$|0,0\rangle_X \mapsto |1\rangle_J \otimes |1\rangle_M \otimes |{+}{+}\rangle_P$$
$$|1,1\rangle_X \mapsto |1\rangle_J \otimes |{-1}\rangle_M \otimes |{+}{+}\rangle_P$$

(because those tensors are in the symmetric subspace, and $\tilde{Z}$ acts by ± 2, respectively), while

$$|0,1\rangle_X = \frac{1}{\sqrt{2}}\frac{|0,1\rangle + |1,0\rangle}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{|0,1\rangle - |1,0\rangle}{\sqrt{2}} \mapsto \frac{1}{\sqrt{2}}|1\rangle_J \otimes |0\rangle_M \otimes |{+}{+}\rangle_P + \frac{1}{\sqrt{2}}|0\rangle_J \otimes |0\rangle_M \otimes |{+}{-}\rangle_P\,,$$
$$|1,0\rangle_X = \frac{1}{\sqrt{2}}\underbrace{\frac{|0,1\rangle + |1,0\rangle}{\sqrt{2}}}_{\in \mathrm{Sym}^2(\mathbb{C}^2)} - \frac{1}{\sqrt{2}}\underbrace{\frac{|0,1\rangle - |1,0\rangle}{\sqrt{2}}}_{\in \Lambda^2(\mathbb{C}^2)} \mapsto \frac{1}{\sqrt{2}}|1\rangle_J \otimes |0\rangle_M \otimes |{+}{+}\rangle_P - \frac{1}{\sqrt{2}}|0\rangle_J \otimes |0\rangle_M \otimes |{+}{-}\rangle_P\,.$$

Exercise. *Can you write down the Schur transform for $n = 3$?*

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