Arithmetic Coding Summary from L7

"Language model": often given by conditional probability distributions:

$$P(x_n | x_1, \dots, x_{n-1}) \quad \text{for } n = 1, 2, \dots, N$$

(x^{n-1})

equivalent

w/ joint distribution

$$P(x^n) = P(x_1) P(x_2|x_1) P(x_3|x_1 x_2) \cdots P(x_n|x^{n-1})$$

↳ see last lecture notes + exercise class

Arithmetic coding:

Input: $x^N \in \mathcal{A}^N$ to compress

Algo:

* $q \leftarrow 0, \ r \leftarrow 1, \ p \leftarrow 1$
* For $n = 1, 2, \dots, N$:
  1. $r \leftarrow q + p\, R(x_n | x_1, \dots, x_{n-1})$
     $q \leftarrow q + p\, Q(x_n | x_1, \dots, x_{n-1})$

     $\sum_{y \le x_n} P(y | x_1 \cdots x_{n-1})$ upper cumulative prob

     lower cumulative prob $\sum_{y < x_n} P(y | x_1 \cdots x_{n-1})$
  2. While $r \le \frac{1}{2}$ or $q \ge \frac{1}{2}$:
     $b \leftarrow \begin{cases} 0 & \text{if } r \le \frac{1}{2} \\ 1 & \text{if } q \ge \frac{1}{2} \end{cases}$
     Write b
     $r \leftarrow 2r - b$
     $q \leftarrow 2q - b$
  3. $p \leftarrow r - q$
* Write $\lceil \log \frac{2}{p} \rceil$ bits of binary expansion of $\frac{q+r}{2}$

Average rate: $\approx \frac{H(X^N)}{N}$ for large N

Joint Entropies (§8)

Joint distribution $P(x,y) \longrightarrow H(XY)$

* marginal distributions: $P(x), P(y) \longrightarrow H(X), H(Y)$
  - $H(X)+H(Y) \geq H(XY)$, $=$ iff X, Y independent — HW 3
* Conditional distributions: $P(y|x)$, $P(x|y)$
  - $H(Y|X=x) = \sum_y P(y|x) \cdot \log \frac{1}{P(y|x)}$ & similarly $H(X|Y=y)$

Conditional entropy:

$$H(Y|X) := \sum_x P(x)\, H(Y|X=x)$$

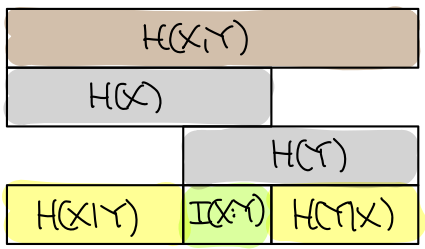
* $H(Y|X) \geq 0$, $=0$ iff $Y = f(X)$ for some function f

  Pf: $=0$ iff $H(Y|X=x)=0$ $\forall x$ iff $\forall x$ (with $P(x)>0$) $\exists y: P(y|x)=1$ □

* $H(Y|X) = H(XY) - H(X)$

  Pf: $H(Y|X) = \sum_{x,y} P(x)P(y|x) \log \frac{1}{P(y|x)}$

  $= \sum_{x,y} P(x,y) \log \frac{P(x)}{P(x,y)} = H(XY) - H(X)$. □



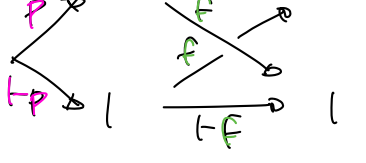
* $H(Y|X) \leq H(Y)$, $=$ iff X, Y independent ⊛

  Pf: eqv to $H(XY) \leq H(X) + H(Y)$? □

* Chain rule: $H(YZ|X) = H(Y|X) + H(Z|XY)$

  Pf: RHS $= H(YX) - H(X) + H(XYZ) - H(XY) =$ LHS. □

ex:



$$H(Y|X) = p \cdot H(\{1-f, f\}) + (1-p) \cdot H(\{f, 1-f\})$$

$$\Rightarrow = H(\{f, 1-f\}) = \begin{cases} 0 & \text{if } f=0 \text{ OR } f=1 \\ 1 & \text{if } f=\frac{1}{2} \end{cases}$$

independent of p?

ex: N = # coin flips of biased coin until 1st heads

$H(N) = ?$ Trick: $X = \begin{cases} 1 & \text{if 1st outcome is heads } (N=1) \\ 0 & \text{otherwise } (N>1) \end{cases}$

$$\Rightarrow H(N) \overset{(*)}{=} H(N,X) = H(X) + H(N|X)$$

$(*)$: $X = f(N)$

$$= \underbrace{H(X)}_{=H(\{p,1-p\})} + p \cdot \underbrace{H(N|X=1)}_{=0 \text{ since } N \neq 1 \text{ if } X=1} + (1-p) \cdot \underbrace{H(N|X=0)}_{=H(N)}$$

$$\Rightarrow H(N) = \frac{H(\{p,1-p\})}{p}$$

Mutual information:

$$I(X:Y) = H(X) + H(Y) - H(X,Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$

* $I(X:Y) \geq 0$, $=0$ iff X, Y independent
* $I(X:Y) \leq H(X), H(Y)$

} reformulations of facts for $H(Y|X), H(Y|X)$ from above

* $I(X:Y) = D(P_{XY} \| Q_{XY})$, where $Q(x,y) = P(x)P(y)$

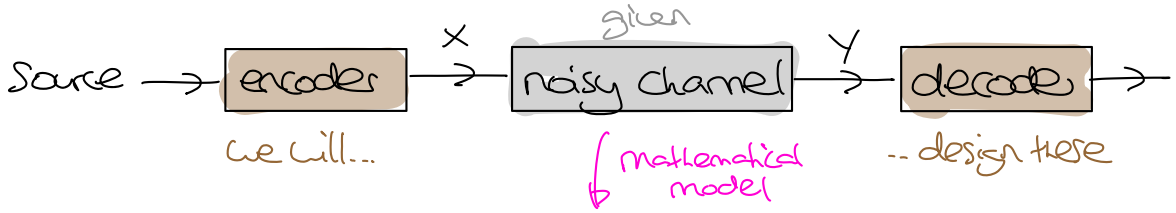
EX CLASS

Recall: Relative entropy: $\frac{0}{0}$ etc!? let's discuss!

$$D(P\|Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)} \quad \in [0, \infty]$$

* $D(P\|Q) < \infty \iff \forall x: Q(x)=0 \Rightarrow P(x)=0$
* Gibbs inequality: $D(P\|Q) \geq 0$, $=0$ iff $P=Q$

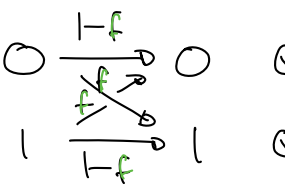
Communicating over Noisy Channels (§9)



(Discrete memoryless) channel: $Q(y|x)$ cond. probability dist.

where $x \in \mathcal{A}_X$ input alphabet, $y \in \mathcal{A}_Y$ output alphabet

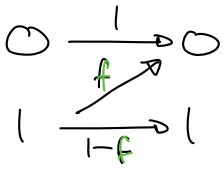
eg. ① Binary symmetric channel: (our old friend)



$Q(0|0) = Q(1|1) = 1-f$

$Q(1|0) = Q(0|1) = f$

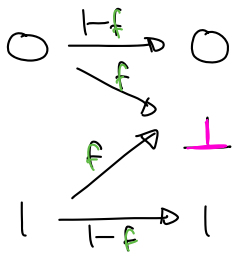
② Binary asymmetric channel: (from HW1)



$Q(0|0)=1,\ Q(1|0)=0$

$Q(0|1)=f,\ Q(1|1)=1-f$

③ Binary erasure channel:



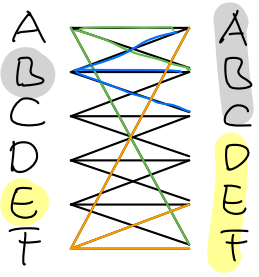
$Q(\perp|0)=Q(\perp|1)=f$

$Q(0|0)=Q(1|1)=1-f$

$Q(1|0)=Q(0|1)=0$

$\mathcal{A}_X=\{0,1\}$ $\mathcal{A}_Y=\{0,1,\perp\}$

④ Noisy typewriter:



$Q(A|A)=Q(B|A)=Q(F|A)=\frac{1}{3}$

$Q(B|B)=Q(C|B)=Q(A|B)=\frac{1}{3}$

$\vdots$

$Q(F|F)=Q(A|F)=Q(E|F)=\frac{1}{3}$

How well can we communicate over each of them?

* If we allow no errors at all:
  - ① ↯ any y could come from either x
  - ② ↯ $y=0$ can come from any x (if sending 0 all the time is not informative)
  - ③ ↯ $y=\perp$ can come from either x
  - ④ 😊 encode $0\mapsto B$, $1\mapsto E$; decode $A,B,C\mapsto 0$, $D,E,F\mapsto 1$

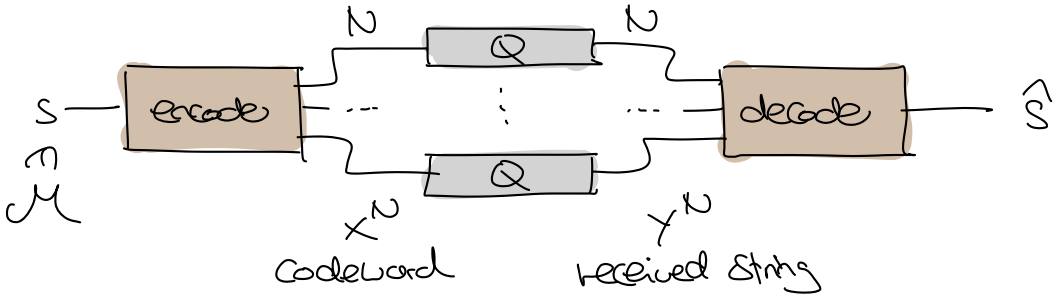
"zero error comm." → EX CLASS

* If we allow error: Can use Bayes' theorem to infer most likely x:

$$P(x|y)=\frac{Q(y|x)\,P(x)}{\sum_z Q(y|z)\,P(z)}$$

assuming x come from some ensemble
↳ Lecture 1 & 2

For reliable communication, consider block encodings:



WANT: $S=\hat{S}$ with high probability

e.g. for S uniform

Rate $R = \frac{\log \#\mathcal{M}}{N}$ bits per channel use

e.g. $R = \frac{\log \#\mathcal{A}}{N}$ for N-fold repetition code

the larger the better

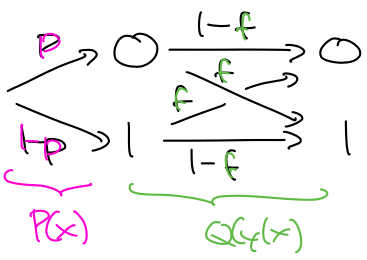
Capacity of a channel $Q(y|x)$ is defined as:

$$C(Q) = \max_{P(x)} I(X:Y)$$

for $P(x,y) = P(x) \cdot Q(y|x)$

Next week we will discuss **Shannon's Noisy Coding Theorem**, which states that $C(Q)$ is the "optimal" rate at which we can communicate "reliably" via the channel $Q(y|x)$.

e.g. for the binary symmetric channel:



* $I(X:Y) = H(Y) - H(Y|X)$
  $= H(Y) - H(\{f, 1-f\})$ (see above)

  ($H(\{f,1-f\})$ indep of p)

* $\max_p H(Y) = 1$ since $P(Y=0) = p(1-f) + (1-p)f = \frac{1}{2}$ if $p = \frac{1}{2}$

$$\Rightarrow C(Q) = \max_p I(X:Y) = 1 - H(\{f,1-f\}) = \begin{cases} 0 & \text{if } f = \frac{1}{2} \\ 1 & \text{if } f = 0 \text{ or } f = 1 \\ \text{inbetween} & \text{otherwise} \end{cases}$$

Intuitive!