

Introduction to Information Theory, Fall 2021

Homework problem set #5

due Dec 5, 2021

Rules: Always explain your solutions carefully. Please hand in the assignment in groups on Canvas. In the werkcollege the TAs can tell you more about how this works.

1. **Joint typicality (1 point):** Let X be random bit with $\Pr(X = 0) = 1/4$, and let Y be the output obtained by sending X through a binary symmetric channel with bit flip probability $f = 1/4$.

(a) Write down the joint distribution $P(x, y)$ and the marginal distributions $P(x)$ and $P(y)$.

(b) Consider the following two options:

• $x^N = 1111010110101111$, $y^N = 1101011110101100$

• $x^N = 1111010110101111$, $y^N = 1011011101100011$

One of the two pairs (x^N, y^N) is in the jointly typical set $J_{16, \epsilon}$ for $\epsilon = 0.1$. Which one?

2. **Entropy inequalities and chain rule (1 point):** In this problem you can practice using entropy inequalities and the chain rule for the conditional entropy. Let X^N be a random string of length N with joint distribution $P(x_1, \dots, x_N)$. Here is a warmup problem:

(a) Show that $H(X^N) \leq H(X_1) + \sum_{i=2}^N H(X_i | X_{i-1}) \leq \sum_{i=1}^N H(X_i)$.

Now let Y^N denote the output of a memoryless channel $Q(y|x)$ when we input the string X^N . Thus, the joint distribution of (X^N, Y^N) is given by $P(x^N, y^N) = P(x^N)Q(y_1|x_1) \cdots Q(y_N|x_N)$.

(b) Show that $H(Y_i | X^N Y^{i-1}) = H(Y_i | X_i)$ for $i = 1, \dots, N$.

(c) Deduce that $I(X^N : Y^N) \leq \sum_{i=1}^N I(X_i : Y_i) \leq N C(Q)$.

Hints: In the exercise class you proved that $H(Z|XY) \leq H(Z|Y)$ for any three random variables. Moreover, equality holds if and only if $X \rightarrow Y \rightarrow Z$ is a Markov chain. Use this in parts (a) and (b). In part (c), start by rewriting the mutual information so that you can apply the chain rule.

3.  **Random codes and typical set decoding (1 point):**

In this problem, you will generate a random code, implement the typical set decoder discussed in class, and study its performance for the binary symmetric channel. To get started, open the notebook at <https://colab.research.google.com/github/amsqi/iit21-homework/blob/master/05-homework.ipynb> and follow the instructions.

Please submit both the notebook **and** a PDF printout, or provide a link to your solution on Colab. You can achieve the maximum score if your solution produces the correct output. We will only have a closer look at your code in case of problems.