Introduction to Information Theory, Fall 2020

Homework problem set #5 **due November 30, 2020**

Rules: Always explain your solutions carefully. Please hand in the assignment in groups on Canvas. In the werkcollege the TAs can tell you more about how this works.

1. **Joint typicality (1 point):** Let X be random bit with $\Pr(X = 0) = 1/4$, and let Y be the output obtained by sending X through a binary symmetric channel with bit flip probability $f = 1/4$.

   (a) Write down the joint distribution $P(x, y)$ and the marginal distributions $P(x)$ and $P(y)$.
   (b) Consider the following two options:
   - $x^N = \texttt{1111010110101111}$, $y^N = \texttt{1101011110101100}$
   - $x^N = \texttt{1111010110101111}$, $y^N = \texttt{1011011101100011}$

   One of the two pairs (x^N, y^N) is in the jointly typical set $J_{16,\varepsilon}$ for $\varepsilon = 0.1$. Which one?

2. **Entropy inequalities and chain rule (1 point):** In this problem you can practice using entropy inequalities and the chain rule for the conditional entropy. Let X^N be a random string of length N with joint distribution $P(x_1, \ldots, x_N)$. Here is a warmup problem:

   (a) Show that $H(X^N) \leqslant H(X_1) + \sum_{i=2}^{N} H(X_i|X_{i-1}) \leqslant \sum_{i=1}^{N} H(X_i)$.

   Now let Y^N denote the output of a memoryless channel $Q(y|x)$ when we input the string X^N. Thus, the joint distribution of (X^N, Y^N) is given by $P(x^N, y^N) = P(x^N)Q(y_1|x_1)\cdots Q(y_N|x_N)$.

   (b) Show that $H(Y_i|X^N Y^{i-1}) = H(Y_i|X_i)$ for $i = 1, \ldots, N$.
   (c) Deduce that $I(X^N : Y^N) \leqslant \sum_{i=1}^{N} I(X_i : Y_i) \leqslant N\,C(Q)$, as claimed in class.

   Hints: In the exercise class you proved that $H(Z|XY) \leqslant H(Z|Y)$ for any three random variables. Moreover, equality holds if and only if $X \to Y \to Z$ is a Markov chain. Use this in parts (a) and (b). In part (c), start by rewriting the mutual information so that you can apply the chain rule.

3. **Random codes and typical set decoding (1 point):**
   In this problem, you will generate a random code, implement the typical set decoder discussed in class, and study its performance for the binary symmetric channel. To get started, open the notebook at https://colab.research.google.com/github/amsqi/iit20-homework/blob/master/05-homework.ipynb and follow the instructions.

   Please submit both the notebook **and** a PDF printout, or provide a link to your solution on Colab. You can achieve the maximum score if your solution produces the correct output. We will only have a closer look at your code in case of problems.