Introduction to Information Theory, Fall 2019

Homework problem set #5 **due October 11, 2019**

Rules: Always explain your solutions carefully. You can work in groups, but must write up your solutions alone. You must submit your solutions before the Friday exercise class (either in person or by email).

1. **Joint typicality (1 point):** Let X be random bit with $\Pr(X = 0) = 1/4$, and let Y be the output obtained by sending X through a binary symmetric channel with bit flip probability $f = 1/4$.

   (a) Write down the joint distribution $P(x, y)$ and the marginal distributions $P(x)$ and $P(y)$.
   (b) Consider the following list:
   - $x^N = \texttt{1111010110101111}$, $y^N = \texttt{1101011110101100}$
   - $x^N = \texttt{1111010110101111}$, $y^N = \texttt{1011011101100011}$

   One of the two pairs (x^N, y^N) is in the jointly typical set $J_{16,\varepsilon}$ for $\varepsilon = 0.1$. Which one?

2. **Entropy inequalities and chain rule (1 point):** In this problem you can practice using entropy inequalities and the chain rule for the conditional entropy. Let X^N be a random string of length N with joint distribution $P(x_1, \ldots, x_N)$. Here is a warmup problem:

   (a) Show that $H(X^N) \leqslant H(X_1) + \sum_{i=2}^N H(X_i|X_{i-1}) \leqslant \sum_{i=1}^N H(X_i)$.

   Now let Y^N denote the output of a memoryless channel $Q(y|x)$ when we input the string X^N. Thus, the joint distribution of (X^N, Y^N) is given by $P(x^N, y^N) = P(x^N)Q(y_1|x_1)\cdots Q(y_N|x_N)$.

   (b) Show that $H(Y_i|X^N Y^{i-1}) = H(Y_i|X_i)$ for $i = 1, \ldots, N$.
   (c) Deduce that $I(X^N : Y^N) \leqslant \sum_{i=1}^N I(X_i : Y_i) \leqslant N\, C(Q)$, as claimed in class.

   Hints: In the exercise class you proved that $H(Z|XY) \leqslant H(Z|Y)$ *for any three random variables. Moreover, equality holds if and only if* $X \to Y \to Z$ *is a Markov chain. Use this in parts (a) and (b). In part (c), start by rewriting the mutual information so that you can apply the chain rule.*

3. **Random codes and typical set decoding (1 point):**
   In this problem, you will generate a random code, implement the typical set decoder discussed in class, and study its performance for the binary symmetric channel. To get started, open the notebook at https://colab.research.google.com/github/amsqi/iit19-homework/blob/master/05-homework.ipynb and follow the instructions.

   As always, please submit your solution as a Python notebook or script, or as a PDF printout. You can score the maximum score if your solution produces the correct output. We will only have a closer look at your code in case of problems.